CALCULUS
Exercise Set 5
Partial Derivatives

1 Domains, Level Curves and Limits

1. Find the domain of the following functions

   (a) $f(x, y) = \sqrt{4 - x^2 - y^2}$

   (b) $f(x, y) = \dfrac{1}{\sqrt{1 - 2x^2 - y^2}}$

   (c) $f(x, y) = \sqrt{\dfrac{1 - x^2}{y^2 - 1}}$

   (d) $f(x, y, z) = \ln(xyz)$

   (e) $f(x, y, z) = \arcsin(\dfrac{1}{x + y + z})$

2. Determine the level curves of the following functions

   (a) $f(x, y) = x + y$

   (b) $f(x, y) = 4 - x^2 - y^2$

   (c) $f(x, y) = x^2 - y^2$

   (d) $f(x, y) = |x| + |y|$

   (e) $f(x, y) = \ln(xy)$

3. Determine the level surfaces of the following functions

   (a) $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

(b) $f(x,y,z) = \dfrac{x^2}{16} + \dfrac{y^2}{4} + \dfrac{z^2}{9}$

(c) $f(x,y,z) = z - x^2 - y^2$

(d) $f(x,y,z) = 2x^2 + z^2$

(e) $f(x,y,z) = x^2 + y^2 - z^2$

4. Find the following limits:

   (a) $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x - y}$

   (b) $\lim_{(x,y)\to(0,0)} xy\dfrac{x^2 - y^2}{x^2 + y^2}$

   (c) $\lim_{(x,y)\to(0,0)} \dfrac{x^3}{x^2 + y^2}$

   (d) $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y^2}{x^2 + y^2}$

   (e) $\lim_{(x,y)\to(0,0)} (x^2 + y^2) \sin \dfrac{1}{x^2 + y^2}$

   (f) $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$

5. Show that the following limits do not exist by using different paths to $(0,0)$:

   (a) $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2 + y^2}$

   (b) $\lim_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2 + y^2}}$

   (c) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{|xy|}$

   (d) $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$

   (e) $\lim_{(x,y)\to(0,0)} \dfrac{x^4 - y^2}{x^4 + y^2}$

   (f) $\lim_{(x,y)\to(0,0)} \dfrac{2x^2 y}{x^4 + y^2}$

(g) $\lim_{(x,y)\to(0,0)} \dfrac{y}{x}(x^4 + y^2)$

2 Partial Derivatives and Differentials

1. Let $f(x, y) = e^{xy^2}$.Check the following identities:

   (a) $f_{xy} = f_{yx}$.

   (b) $f_{xxy} = f_{xyx} = f_{yxx}$.

2. *Laplace Equation* The equation involving the partial derivatives of a function $f(x, y, z)$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

   is known as the Laplace equation. Check whether the following functions satisfy the Laplace equation or not.

   (a) $f(x, y, z) = (x - a)^2 + (y - b)^2 - 2(z - c)^2$.

   (b) $f(x, y, z) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$.

   (c) $f(x, y, z) = \sin ax \sin by \cosh cz$.

3. *Wave Equation* The equation involving the partial derivatives of a function $f(t, x)$

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

   is known as the wave equation. Check whether the following functions satisfy the wave equation or not.

   (a) $f(x, t) = a \sin(x - ct) + b \sin(x + ct)$.

   (b) $f(x, t) = \sin act \sin bx$.

   (c) $f(x, t) = \ln(ax + act)(bx - bct)$.

4. Find the linearization L of f at the point P:

   (a) $f(x, y) = 100 - 20x^2 - 30y^2; \quad P = (1, 1)$

   (b) $f(x, y) = x^2 - xy + y^2; \quad P = (1, 2)$.

(c) $f(x, y) = e^{-(x^2+y^2)}$ and $P = (1/2, 1/3)$.

(d) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$; $\quad P = (0, 1, 0)$.

(e) $f(x, y, z) = e^{-x} \sin(y + z)$; $\quad P = (0, 0, \pi)$.

5. The resistance of three resistors R_1, R_2 and R_3 connected in parallel is given by the expression

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

(a) Show that the differential dR is given by

$$dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2 + \left(\frac{R}{R_3}\right)^2 dR_3.$$

(b) Let $R_1 = 100$, $R_2 = 200$ and $R_3 = 250$ ohms. Estimate the error in the measurement of R if errors in the measurements of the resistors is $\Delta R_1 = \pm 5, \Delta R_2 = \pm 6$ and $\Delta R_3 = \pm 10$.

(c) Estimate the maximum percentage error possible in the measurement of the resistors R_1, R_2 and R_3 if the maximum error in the total resistance R is to be less than 5%.

3 Chain Rule and Implicit Differentiation

1. Let $f(x, y) = \cos(x^2 y)$, $x(t) = e^{2t}$ and $y(t) = \tan 3t$.

   (a) Find the derivative of $F(t) = f(x(t), y(t))$ with respect to t by substitution and differentiation.

   (b) Find the same derivative directly using the chain rule.

2. Let $f(x, y, z) = \sin(x + y + z)$, $x(t) = e^{2t}$, $y(t) = \ln t$ and $y(t) = t^{3/4}$.

   (a) Find the derivative of $F(t) = f(x(t), y(t), z(t))$ with respect to t by substitution and differentiation.

   (b) Find the same derivative directly using the chain rule.

3. *Laplace Equation in polar coordinates* Let $z = f(x, y)$, $x = r\cos\theta$ and $y = r\sin\theta$. show the following expression

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}.$$

4. Find values of m such that $z = f(y + mx)$ is a solution of the partial differential equation

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial y \partial x} + c\frac{\partial^2 z}{\partial y^2} = 0.$$

5. Given the equation

$$e^z \sin(x + y) + e^y \sin(x + z) + e^x \sin(y + z) = 0$$

find the first and second derivatives of z with respect to x and y at $(\pi, 0, \pi/2)$.

6. At what points does the equation

$$x^4 + y^4 + z^2 + 2xz - 2yz + 5 = 0$$

define a function $z = g(x, y)$ implicitly? Find its partial derivatives at those points

7. Given the system of equations

$$\begin{aligned} u^2 - v + x^2 + y^2 &= 0 \\ u + v^2 - 2xy &= 0 \end{aligned}$$

find the first and second derivatives of u and v with respect to x and y.