CALCULUS
Exercise Set 2
Integration

1 Basic Indefinite Integrals

1. $\int 0 \; dx = C$
2. $\int x^n \; dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$
3. $\int \frac{1}{x} \; dx = \ln |x| + C$
4. $\int \sin x \; dx = -\cos x + C$
5. $\int \cos x \; dx = \sin x + C$
6. $\displaystyle\int \frac{dx}{\cos^2 x} = \tan x + C$
7. $\displaystyle\int \frac{dx}{\sin^2 x} = -\cot x + C$
8. $\int \sec x \tan x \; dx = \sec x + C$
9. $\int \csc x \cot x \; dx = -\csc x + C$
10. $\int e^x \; dx = e^x + C$
11. $\int a^x \; dx = \frac{a^x}{\ln a} + C \qquad (a > 0, \, a \neq 1)$
12. $\int \sinh x \; dx = \cosh x + C$
13. $\int \cosh x \; dx = \sinh x + C$

14. $\int \frac{dx}{\cosh^2 x} = \tanh x + C$

15. $\int \frac{dx}{\sinh^2 x} = -\coth x + C$

16. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

17. $\int \frac{dx}{1-x^2} = \tanh^{-1} x + C$

18. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

19. $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$

20. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$

21. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} |x| + C$

2 Indefinite Integral

1. Evaluate the following integrals by completing the square:

   (a) $\int \frac{dx}{\sqrt{6x-x^2}}$

   (b) $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

2. Evaluate the following integrals using the trigonometric identities

$$\begin{aligned} \sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2} \end{aligned}$$

   (a) $\int \sin^2 x \; dx$

(b) $\int \cos^2 x \, dx$

3. Evaluate the following integrals

   (a) $\int \tan x \, dx$ (use the substitution $\cos x = u$)

   (b) $\int \sec x \, dx$ (multiply and divide by $\sec x + \tan x$)

   (c) $\int \cos^3 x \, dx$ (substitute $\cos^2 x = 1 - \sin^2 x$ and use the change of variables $\sin x = u$)

   (d) $\int \sin^5 x \, dx$ (substitute $\sin^2 x = 1 - \cos^2 x$ and use the change of variables $\cos x = u$)

4. Evaluate the following integrals using integration by parts

   (a) $\int xe^{ax} \, dx$

   (b) $\int x \cos ax \, dx$

   (c) $\int \ln x \, dx$

   (d) $\int \arcsin x \, dx$

   (e) $\int \arctan x \, dx$

5. A proper rational function (i.e. degree of the numerator is less than the degree of the denominator) can be decomposed in *partial fractions* in the following way: For each linear fact linear factor write a fraction of the form

$$\frac{A}{x-a}$$

For each factor of the form $(x-a)^n$ write an expression of the form

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_p}{(x-a)^n}.$$

For each irreducible quadratic factor $x^2 + px + q = (x-a)^2 + b^2$ write a fraction of the form

$$\frac{Ax+B}{(x-a)^2+b^2}$$

For each factor of the form $[(x-a)^2 + b^2]^n$ write an expression of the form

$$\frac{A_1x+B_1}{(x-a)^2+b^2} + \frac{A_2x+B_2}{[(x-a)^2+b^2]^2} + \cdots + \frac{A_nx+B_n}{[(x-a)^2+b^2]^n}.$$

Evaluate the following integrals using partial fraction decomposition:

(a) $\int \frac{dx}{(x-1)(x+2)}$

(b) $\int \frac{dx}{x(x+1)^3}$

(c) $\int \frac{dx}{(x-1)(x^2+x+1)}$

(d) $\int \frac{(2x+1)\,dx}{x(x^2+1)^2}$

3 Initial Value Problems

1. Solve the following initial value problems

   (a) $y' = x^2$, $y(0) = -1$

   (b) $y' = \sin 2x$, $y(\pi) = 1$

   (c) $y' = 3x^{-2/3}$, $y(-1) = 2$

2. Solve the following initial value problems by the method of separation of variables

   (a) $y' = 5y^{3/2}$, $y(0) = 9$

   (b) $y' = y/x$, $y(2) = -1$

   (c) $x^2y' - y - xy = 0, y(-1) = 5$

3. Find the curve that goes through the point $(4, 9)$ and whose slope at that point is $3\sqrt{x}$.

4. Find the curve that goes through the point $(1, 3)$ and whose slope at that point is $-x/y$.

5. You are driving at a speed of 120 Km/h when you see an accident ahead of you blocking the highway and slam on your brakes. Determine the constant deceleration you need to apply to bring the car to a complete stop in 100 meters.

6. Radioactive substances decay at a rate which is proportional to the amount of such substance at the given time. Find the time needed for a radioactive substance to be reduced in half and show that this time does not depend on the initial amount present.

4 Definite Integral

1. Evaluate the following integrals

   (a) $\int_0^1 \sqrt{1-x^2}\,dx$
   (b) $\int_0^2 |1-x^2|\,dx$
   (c) $\int_0^{100\pi} \sqrt{1-\cos 2x}\,dx$
   (d) $\int_0^1 \lfloor 3x \rfloor\,dx$

2. Evaluate the following integrals by substitution

   (a) $\displaystyle\int_0^{\ln 2} \sqrt{e^x - 1}\,dx$
   (b) $\displaystyle\int_{-1}^{1} \frac{1+x^2}{1+x^4}\,dx \qquad (x - \tfrac{1}{x} = u).$

3. Evaluate the following integrals by parts

   (a) $\displaystyle\int_0^{\pi} x\cos x\,dx$
   (b) $\displaystyle\int_0^{1} x^2 e^{-x}\,dx$

4. Find the average value of the function

$$f(x) = \sqrt{1-x^2}$$

   on the interval $[0,1]$.

5. The voltage of a 60 cycles per second household mains is given by

$$v = V_P \sin(120\pi t)$$

   where V_P is the peak voltage.

   (a) Find the average value of v over a half cycle. What is the average value over a whole cycle?
   (b) Find the average value of v^2. Its square root is called the effective voltage and it measures the power dissipated per unit of resistance, what is the peak voltage when the the effective voltage is 220 volts?

6. *Error Function* The function $f(t) = e^{-t^2}$, being continuous, admits an antiderivative but this derivative can not be expressed by an algebraic expression. The function

$$F(x) = \frac{2}{\sqrt{\pi}} \int_0^x f(t)\, dt.$$

is called the error function. Find:

(a) Symmetries of F. and axis intersections.

(b) Intervals on which F increases, the intervals on which it decreases and its extreme values.

(c) Intervals on which F is concave up, the intervals on which it is concave down its inflection points.

(d) Sketch the graph F.

5 Areas and Volumes

1. Find the area enclosed by the curves

$$y = \sin x \text{ and } y = \sin 2x$$

between $x = 0$ and $x = \frac{\pi}{2}$.

2. Find the area of the points inside a square of side L which are closer to the center of the square than to the edge.

3. Consider the region enclosed by the two parabolas

$$y^2 = 4px \text{ and } x^2 = 4py$$

with $p > 0$. Find the volume of the solid generated by revolving this region around:

(a) The x-axis.

(b) The y-axis.

(c) The line $x = -p$

(d) The line $y = 4p$.

4. Consider a sphere of radius R.

   (a) Find the volume of a cap of height h.

   (b) Find the volume of the portion of the sphere remaining after drilling a hole of radius $r < R$ along one of its diameters.

5. Find the volume of the wedge cut out of a cylindrical tree by making two cuts: one perpendicular to the axis and another one at an angle of 45 degrees.

6. Find the volume of the solid whose base is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   and its cross sections perpendicular to the x-axis are:

   (a) Equilateral triangles.

   (b) Squares.

   (c) Isosceles right triangles with the hypotenuse on the xy-plane.

6 Improper Integrals

1. Evaluate the following improper integrals

   (a) $\int_0^\infty \frac{dx}{1+x^2}$

   (b) $\int_0^\infty xe^{-x^2}\,dx$

   (c) $\int_\pi^\infty \cos^2 x\,dx$

   (d) $\int_{-\infty}^0 e^{2x}\,dx$

2. Evaluate the following improper integrals

   (a) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

(b) $\int_0^1 x \ln x \, dx$

(c) $\int_0^{\pi/2} \sec x \, dx$

(d) $\int_{-3}^3 \frac{1}{x(x+1)} \, dx$

3. Prove that
$$\int_1^\infty \frac{1}{x^p} \, dx = \frac{1}{p-1}$$
if $p > 1$ and that the integral diverges for $p \leq 1$.

4. Prove that
$$\int_0^1 \frac{1}{x^p} \, dx = \frac{1}{1-p}$$
if $p < 1$ and that the integral diverges for $p \geq 1$.

5. Evaluate the following improper integrals

(a)
$$\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}} \, dx$$

(b)
$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)} \, dx$$