CALCULUS
Lab Practice Set 6

1 New Maple Commands

- **plot**($[x(t), y(t), t = a..b]$) graphs the curve given by the parametric equations $x(t), y(t)$ from $t = a$ to $t = b$

- **polarplot**($r(\theta)$, $\theta = a..b$); graphs the curve given by the polar equation $r = r(\theta)$ between $\theta = a$ and $\theta = b$

2 Exercises

1. A particle moves along a plane curve whose parametric equations are

$$x(t) = e^{-t}\cos t \ , \ y(t) = e^{-t}\sin t \ , \ t \geq 0.$$

   (a) Graph x and y as functions of t.

   (b) Graph the path traced by the particle in the xy-plane.

   (c) Find the direction of motion at $t = 5$.

   (d) Find the speed of the particle as a function of t and graph this function.

   (e) Find the distance travelled by the particle from $t = 0$ to $t = 5$.

   (f) Repeat the above questions for the following equations

      i. $x(t) = t$; $y(t) = 18 - 0.5t^2$

      ii. $x(t) = t$; $y(t) = 1/(1 + t)$

      iii. $x(t) = e^{3^t} + \sin 2t$; $y(t) = e^{3t} + \cos t^2$

iv. $x(t) = 1 + \ln t$; $y(t) = t \ln t$

v. $x(t) = t/(1+t^3)$; $y(t) = t^2/(1+t^3)$

2. The path traced by a point P on the circumference of a wheel of radius a rolling without slipping along a horizontal straight line is called a *cycloid*.

   (a) Prove that the parametric equations of the cycloid are

   $$\begin{aligned} x &= at - a\sin t \\ y &= a - a\cos t \end{aligned}$$

   where t is the angle rolled by the point P around the center of the circle assuming that initially the center of the circle is on the y-axis and the point P is touching the x-axis.

   (b) Plot the graph of the curve for $a = 1$.

   (c) Find the magnitude of the velocity (speed) and acceleration at each point on the path. Plot them as a function of the parameter used to describe the cycloid.

3. The path traced out by a point P on a radius of a wheel rolling without slipping along a horizontal straight line is called a *trochoid*. Let a be the radius of the wheel and b the distance from the point P to the center of the wheel.

   (a) Prove that the parametric equations of the trochoid are

   $$\begin{aligned} x &= at - b\sin t \\ y &= a - b\cos t \end{aligned}$$

   where t is the angle rolled by the point P around the center of the circle assuming that initially the center of the circle is on the y-axis and the point P is touching the x-axis.

   (b) Plot the graph of the curve for different values of a and b.Observe that for $a > b$ the trochoid has no vertical lines and for $a < b$ the curve has two vertical lines per cycle.

4. The path traced out by a point P on the circumference of a small circle of radius a rolling without slipping around the outside of a large circle of radius b is called an *epicycloid*.

   (a) Prove that the parametric equations of the epicycloid are

   $$\begin{aligned} x &= (a+b)\cos t - a\cos\frac{a+b}{a}t \\ y &= (a+b)\sin t - a\sin\frac{a+b}{a}t \end{aligned}$$

   where t is the angle rotated by the ray that joins the origin and the center of the small circle assuming that initially the center of the small circle lies on the x-axis.

   (b) Plot the graph of the curve for $b = 1$ and $a = 1$,2, 3 and 4 respectively.

5. The path traced out by a point P on the circumference of a wheel of radius b rolling without slipping inside a circle of radius a, $a > b$, is called a *hypocycloid*.

   (a) Prove that the parametric equations of the hypocycloid are

   $$\begin{aligned} x &= (a-b)\cos t + b\cos\frac{a-b}{b}t \\ y &= (a-b)\sin t - b\sin\frac{a-b}{b}t \end{aligned}$$

   where t is the angle rotated by the ray that joins the origin and the center of the small circle assuming that initially the center of the small circle lies on the x-axis.

   (b) Plot the graph of the hypocycloid for $a = 1$; $b = 1/3$ and for $a = 1$; $b = 1/2$. How many cusps does the hypocycloid have in each case?

   (c) Plot the graph of the hypocycloid for $a = 1$; $b = 4/5$ and for $a = 1$; $b = 3/5$. How many cusps does the hypocycloid have in each case?

(d) If $b = a/4$ the hypocycloid has four cusps and is called *astroid*. Show that its parametric equations are

$$\begin{aligned} x(t) &= a\cos^3 t \\ y(t) &= a\sin^3 t. \end{aligned}$$

6. Find the length of the ellipse

$$x(t) = 4\cos t,\ y(t) = 3\sin t,\ 0 \le t \le 2\pi.$$

Write its length as an integral of the form

$$\int_0^{2\pi} \sqrt{1 - k^2\cos^2 t}\, dt$$

called *elliptic integral*. The integrand does not admit an elementary antiderivative. Approximate its value to four decimal places.

7. *Helixes*

(a) Plot a graph of the space curve defined by the equations

$$x(t) = \cos t,\ y(t) = \sin t,\ z(t) = t$$

and find its length for $0 \le t \le 2\pi$.

(b) Repeat in the following cases:

i. $x(t) = \cos 5t$, $y(t) = \sin 5t$, $z(t) = t$.
ii. $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 3t$.
iii. $x(t) = t\cos 3t$, $y(t) = t\sin 3t$, $z(t) = t$.

8. *Lissajous curves* This is a family of plane curves defined by the parametric equations

$$x(t) = \sin mt,\ y(t) = \cos nt,\ 0 \le t \le 2\pi.$$

and they represent complex harmonic motion. Plot a few of these curves when m and n are consecutive Fibonacci numbers: 1, 1, 2, 3, 5, 8, $\cdots$.

9. *Limaçons* These are curves defined in polar coordinates by the equation

$$r = 1 + a\cos\theta$$

Plot a graph of the following limaçons:

(a) $r = 1 + \cos\theta$ *Cardiod.*

(b) $r = 1 + 2\cos\theta$ *Limaçon with an inner loop*

(c) $r = 1 + 0.75\cos\theta$ *Limaçon with a dimple*

(d) $r = 1 + 0.5\cos\theta$ *Convex limaçon*

10. *Roses* The plane curves represented in polar coordinates by the equation

$$r = a\cos(n\theta)$$

have the shape of a petaled flower with the number of petals depending on n.

(a) Plot a graph of these curves for $a = 1$ and $n = 1, 2, 3, 4$ respectively.

(b) Plot a graph of these curves for $n = 3$ and $a = 0.5, 1$, 3 respectively.

(c) Plot a graph of these curves for $a = 1$ and $n = 1/4, 1/2, 3/4$ respectively.

(d) Plot a graph of these curves for $a = 1$ and $n = \sqrt{2}, \pi$ respectively.

(e) In view of the above results answer the following questions:

i. What is the length of each petal?

ii. How may petals are there if n is odd?

iii. How may petals are there if n is even?

iv. How may petals are there if n is rational?

v. What happens when n is irrational?

11. *Spirals* Plot a graph of the given polar curve

(a) $r = e^{\theta/10}$ *Logarithmic spiral.*

(b) $r\theta = 6$ *Hyperbolic spiral.*

(c) $r = 3\theta$ *Arquimedian spiral.*

(d) $r^2\theta = 1$ *Lituus.*

12. Plot a graph of the given polar curve

(a) $r = 1 - 2\sin 3\theta$ *A rose within a rose.*

(b) $r = 1 - 2\sin 4\theta$ *A rose with petals of different size.*

(c) $r = 1 + 2\sin(\theta/2)$ *Freeth's Nefroid.*

(d) $r = \sin\theta/\theta$ *Cochleoid.*

(e) $r = (\cos\theta)(\cos^4\theta - 1)$ *Folium.*

(f) $r = 2 - \csc\theta$ *Concoid.*