CALCULUS
Lab set Practice 5

1 New Maple Commands

- **plot**([*seq(*[*n,a(n)*]*,n=1..N)*], style=point) Plots the sequence $a(n)$
- **sum**(*a(n),n=1..N*) Adds the N first terms of the sequence $a(n)$
- **sum**(*a(n),n=1..infinity*) Finds the sum of the series $\sum a(n)$
- **Sum**(*a(n),n=1..infinity*) Symbol for the sum of the series $\sum a(n)$)
- **taylor**(*f(x),x=a,N*) N first terms in the Taylor series expansion of $f(x)$ around a
- **convert**(taylor(*f(x),x=a,N*),polynom) Taylor's polynomial of order N

2 Exercises

1. Given the sequence $a_n = 2^{-n}$,

   (a) Find its first 20 terms and graph the sequence plotting the corresponding points (n, a_n).

   (b) Does the sequence converge? If so, find its limit.

   (c) If the sequence converges find an N such that for all $n > N$, a_n approaches the limit with an error less than ± 0.01.

   (d) Repeat the above questions for the following sequences:

      i. $a_n = 3^{1/n}$.

ii. $a_n = n^{1/n}$.

iii. $a_n = \dfrac{\ln n}{n}$.

iv. $a_n = \dfrac{5^n}{n!}$.

v. $a_n = \left(1 + \dfrac{1}{n}\right)^n$.

vi. $a_n = \dfrac{5^n}{7n}$.

vii. $a_n = (-1)^{n+1}\dfrac{n-1}{n}$.

2. You want to deposit \$10,000 on a bank account that pays interest at an annual percentage rate r. Interest is compounded n times per year.

   (a) Find a formula that gives the accumulated amount at the end of the year.

   (b) Find the accumulated amount when interest is compounded yearly ($n = 1$), semesterly ($n = 2$), quarterly ($n = 4$), monthly ($n = 12$), weekly ($n = 52$), daily ($n = 365$) and hourly ($n = 8.760$).

   (c) What would the accumulated amount be if interest were compounded continuously, $n \to +\infty$?

3. *Fibonacci Sequence* This sequence is defined by the recursive formula

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$ and the initial values $F_0 = F_1 = 1$.

   (a) Find F_n for $n = 0, \ldots, 10$ and graph the sequence plotting the points (n, F_n).

   (b) Does the Fibonacci sequence converge or diverge?

   (c) Evaluate the ratios $R_n = F_{n+1}/F_n$ and graph the sequence of these ratios plotting the points (n, R_n).

   (d) Find a recursive formula to express this sequence of ratios, show that it converges and find its limit. This limit is known as the *Golden Ratio.*

4. *Geometric Series* A series of the form

$$1 + r + r^2 + \cdots + r^n + \cdots$$

is called a geometric series of ratio r. Let $r = 1/3$,

(a) Evaluate the sequence of partial sums for $n = 0, \cdots, 20$ and draw a point plot of this sequence.

(b) Conjecture whether the series converges or not, and if it does find its sum.

(c) Repeat for $r = -2, -3/2, -1, -1/2, 1/2, 1, 3/2, 2$.

(d) From these plots, conjecture for what values of r the series converges and find a formula for its sum.

5. *Harmonic Series* The series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is known as the harmonic series. By application of the integral test we know that this series diverges. The following experiment shows that the partial sums grow very slowly, so slowly that observing a point plot of the partial sums gives the impression that the series converges.

(a) Evaluate the sequence of partial sums of the harmonic series for $n = 1 \cdots 100$ and draw a point plot of this sequence.

(b) Imagine that you started adding the terms of this series the year the universe was formed, 13 billion years ago, and added a new term every second. How large would the partial sum be today?

6. Given the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

(a) Evaluate the sequence of partial sums for $n = 0, \cdots, 20$ and draw a point plot of this sequence.

(b) Conjecture whether the series converges or not, and prove its convergence or divergence.

(c) If the series converges, find its sum within 0.001.

(d) Repeat the above questions for the following series,

i. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$.

ii. $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

iii. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/2}}$.

iv. $\sum_{n=1}^{\infty} \frac{1}{n^{!/2}}$.

v. $\sum_{n=1}^{\infty} \frac{1}{n!}$.

vi. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

vii. $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$.

viii. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$.

7. Given the series expansion

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

(a) Find the radius of convergence.

(b) Draw a multiplot showing the partial sum $S_n(x)$ and the sum of the series for $n = 0, \cdots, 20$. Observe how the graphs of the partial sums approach the limit within the interval of convergence.

(c) Repeat with the series expansions

i. $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1} x^n$.

ii. $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$.

8. Let $f(x) = e^x$.

(a) Find the first 20 Taylor polynomials of the function $f(x)$.

(b) Draw a multiplot showing the Taylor polynomial $P_n(x)$ and the function $f(x)$ for $n = 0, \cdots, 20$.

(c) Find the radius of convergence of the Taylor series.

(d) Repeat for,

i. $f(x) = \sin x$.

ii. $f(x) = \cos x$.

9. Let $f(x) = e^x$.

(a) Draw a multiplot of the graph of $f(x)$ and the graph of its fifth-degree Taylor polynomial $P_5(x)$.

(b) Find the maximum error made on the interval $[0, 1]$ when the function $f(x)$ is replaced by its Taylor polynomial $P_5(x)$.

(c) Find a Taylor polynomial that approaches the function $f(x)$ within 0.001 on the interval $[0, 1]$.

(d) Repeat the above questions with the function $\ln(1 + x)$.