Multiple Integrals

1 Double Integrals

1.1 Double Integrals over Rectangles

Let $f(x,y)$ be a function defined on the rectangle $R : a \leq x \leq b,\ c \leq y \leq d$. In order to define the integral of f we begin by dividing the intervals $[a,b]$ and $[c,d]$ into n and m subintervals

$$\begin{aligned} a &= x_0 < x_1 < \cdots < x_n = b \\ c &= y_0 < y_1 < \cdots < y_m = d. \end{aligned}$$

This results in a partition Π of the rectangle R into $n \times m$ smaller subrectangles

$$R_{ij} : x_{i-1} \leq x \leq x_i,\ \ y_{j-1} \leq y \leq y_j$$

of area

$$\Delta A_{ij} = \Delta x_i \Delta y_j = (x_i - x_{i-1})(y_j - y_{j-1}).$$

The longest diagonal of all the rectangles R_{ij}, i.e. the number

$$\|\Pi\| = \max\left(\sqrt{(\Delta x_i)^2 + (\Delta y_j)^2}\right)$$

is called the norm of the partition.

Now we arbitrarily choose a point (ξ_i, η_j) in each subrectangle R_{ij} and form the sum

$$\sum_{i,j} f(\xi_i, \eta_j)\, \Delta A_{ij} = \sum_{i,j} f(\xi_i, \eta_j)\, \Delta x_i \Delta y_j.$$

If the function f is continuous and we refine the partition so that its norm $\|\Pi\|$ goes to zero, the sums approach a number independently of the choice of the points (ξ_i, η_j). This number is called the double integral of the function f over R and is denoted by

$$\iint_R f(x,y)\, dx\, dy = \lim_{\|\Pi\| \to 0} \sum_{i,j} f(\xi_i, \eta_j)\, \Delta x_i \Delta y_j.$$

This limit can still exist under less restrictive conditions. For instance if f is continuous except at a finite number of points or continuous arcs.

When f is positive each product $f(\xi_i, \eta_j)\, \Delta x_i \Delta y_j$ represents the volume of a vertical rectangular box with base the rectangle R_{ij} and height $f(\xi_i, \eta_j)$. In this case the above sums represent an approximation of the volume under the graph of f and above the region R and the double integral the exact volume.

1.2 Iterated Integrals

In the preceding section we have defined the double integral of a continuous function $f(x, y)$ over a rectangle $R: a \leq x \leq b$, $c \leq y \leq d$. Now we are going to show a method for evaluating this integral.

If we hold the variable x constant and integrate the function $f(x, y)$ respect to the variable y from $y = c$ to $y = d$ we obtain the function of x

$$F(x) = \int_c^d f(x, y)\, dy.$$

Integrating this new function $F(x)$ from $x = a$ to $x = b$ gives the iterated integral

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx.$$

Similarly, holding the variable y constant and integrating the function $f(x, y)$ respect to the variable x from $x = a$ to $x = b$ we obtain the function of y

$$G(y) = \int_a^b f(x, y)\, dx.$$

Integrating this new function $G(y)$ from $y = c$ to $y = d$ gives the iterated integral

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

It turns out that these two iterated integrals provide the same result and that this value is the value of the double integral of f over the rectangle R.

Theorem 1 (Fubini's Theorem) *If $f(x, y)$ is a continuous function on the rectangle $R: a \leq x \leq b$, $c \leq y \leq d$, then*

$$\begin{aligned} \iint_R f(x, y)\, dx\, dy &= \int_a^b \int_c^d f(x, y)\, dy\, dx \\ &= \int_c^d \int_a^b f(x, y)\, dx\, dy. \end{aligned}$$

Example 2 *Calculate the double integral of $f(x, y) = x^3 y^2$ over the rectangle $R: 0 \leq x \leq 2, -1 \leq y \leq 1$.*

Solution By Fubini's theorem

$$\begin{aligned} \iint_R x^3 y^2\, dx\, dy &= \int_0^2 \int_{-1}^1 x^3 y^2\, dy\, dx \\ &= \int_0^2 \left(\int_{-1}^1 x^3 y^2\, dy \right) dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^2 \left(\frac{1}{3}x^3y^3\right)\Big|_{-1}^{1} \, dx \\
&= \int_0^2 \frac{2}{3}x^3 \, dx \\
&= \left(\frac{1}{6}x^4\right)\Big|_0^2 \\
&= \frac{8}{3}.
\end{aligned}
$$

Reversing the order integration gives the same result

$$
\begin{aligned}
\iint_R x^3y^2 \, dx \, dy &= \int_{-1}^1 \int_0^2 x^3y^2 \, dx \, dy \\
&= \int_{-1}^1 \left(\int_0^2 x^3y^2 \, dx\right) dy \\
&= \int_{-1}^1 \left(\frac{1}{4}x^4y^2\right)\Big|_0^2 \, dy \\
&= \int_{-1}^1 4y^2 \, dy \\
&= \left(\frac{4}{3}y^3\right)\Big|_{-1}^1 \\
&= \frac{8}{3}.
\end{aligned}
$$

1.3 Double Integrals Over Bounded Regions

In this section we extend the definition of double integrals over rectangles to more general regions. Let D be a bounded plane region whose boundary is made up of a finite number of smooth arcs of the form $y = \varphi(x)$ or $x = \psi(y)$. To define the double integral of the function $f(x, y)$ defined on D we choose a rectangle $R : a \leq x \leq b$, $c \leq y \leq d$ such that $D \subset R$ and define an extension of $f^{\,*}$ to the whole rectangle R by

$$
f^*(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \in R - D \end{cases} .
$$

If f is continuous, the function f^* may not be continuous but the double integral of f^* over R still exists and the double integral of f over D is defined by

$$
\begin{aligned}
\iint_D f(x, y) \, dx \, dy &= \iint_R f^*(x, y) \, dx \, dy \\
&= \lim_{\|\Pi\| \to 0} \sum_{i,j} f^*\left(\xi_i, \eta_j\right) \Delta x_i \Delta y_j.
\end{aligned}
$$

When f is positive the double integral over D represents the volume of the solid bounded above by the graph of f and below by the region D. In particular if $f(x, y) = 1$ the double integral gives the area of the region D

$$A = \iint_D 1\, dx\, dy$$

To calculate this integral we can resort to iterated integrals based on the following extension of the Fubini's Theorem

Theorem 3 (Extension of Fubini's Theorem) *Let $f(x, y)$ be a continuous function on a region D*

1. If $D : a \leq x \leq b$, $\varphi_1(x) \leq y \leq \varphi_2(x)$, with $\varphi_1(x)$ and $\varphi_2(x)$ continuously differentiable on the interval $[a, b]$, then

$$\iint_D f(x, y)\, dx\, dy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy\, dx.$$

2. If $D : c \leq y \leq d$, $\psi_1(y) \leq x \leq \psi_2(y)$, with $\psi_1(x)$ and $\psi_2(x)$ continuously differentiable on the interval $[c, d]$, then

$$\iint_D f(x, y)\, dx\, dy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx\, dy.$$

Example 4 *Calculate the double integral of $f(x, y) = xy$ over the region D : $0 \leq x \leq 2, -x \leq y \leq \sqrt{x}$.*

Solution By Fubini's theorem

$$\begin{aligned}
\iint_D xy\, dx\, dy &= \int_0^2 \int_{-x}^{\sqrt{x}} xy\, dy\, dx \\
&= \int_0^2 \left(\int_{-x}^{\sqrt{x}} xy\, dy \right) dx \\
&= \int_0^2 \left(\frac{1}{2} xy^2 \right) \Big|_{-x}^{\sqrt{x}} dx \\
&= \int_0^2 \frac{1}{2} \left(x^2 - x^3 \right) dx \\
&= \left(\frac{1}{6} x^3 - \frac{1}{8} x^4 \right) \Big|_0^2 \\
&= -\frac{2}{3}.
\end{aligned}$$

Example 5 *Find the volume of the solid bounded above by the plane $z = 2x$ and below by the region enclosed by the parabola $y^2 = 1 - x$ and the axis $x = 0$.*

Solution The volume is given by the double integral

$$\iint_D 2x\,dx\,dy$$

where D is the region $0 \le x \le 1-y^2$, $-1 \le y \le 1$.By Fubini's theorem

$$\begin{aligned}
\iint_D 2x\,dx\,dy &= \int_{-1}^{1}\int_0^{1-y^2} 2x\,dx\,dy \\
&= \int_{-1}^{1}\left(\int_0^{1-y^2} 2x\,dx\right) dy \\
&= \int_{-1}^{1} x^2\Big|_0^{1-y^2}\,dy \\
&= \int_{-1}^{1}\left(1-2y^2+y^4\right) dy \\
&= 2\int_0^1\left(1-2y^2+y^4\right) dy \\
&= 2\left(y-\frac{2}{3}y^3+\frac{1}{5}y^4\right)\Big|_0^1 \\
&= \frac{16}{15}.
\end{aligned}$$

1.4 Double Integrals in Polar Coordinates

Sometimes plane regions are better described using polar coordinates than rectangular coordinates. Let D be a region in the xy-plane, described in polar coordinates by Γ; $\alpha \le \theta \le \beta$, $a(\theta) \le r \le b(\theta)$, and $f(x,y)$ a function defined on D

In this case instead of partitioning the region into small rectangles R_{ij} it can be more appropriate to use polar rectangles Γ_{ij}, i.e. cells bounded by the rays $\theta = \theta_{j-1}$ and $\theta = \theta_j$ and by the circles $r = r_{i-1}$ and $r = r_i$. The area of each cell is given by

$$\begin{aligned}
\Delta A_{ij} &= \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta_j \\
&= \frac{1}{2}\left(r_i + r_{i-1}\right)\Delta r_i \Delta\theta_j
\end{aligned}$$

In the limit the infinitesimal area element is given by

$$dA = r\,dr\,d\theta$$

This leads to the following formula to calculate the double integral of a function f over D using polar coordinates

$$\iint_D f(x,y)\,dx\,dy = \iint_\Gamma f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

$$= \int_{\alpha}^{\beta} \int_{a(\theta)}^{b(\theta)} f(r\cos\theta, r\sin\theta) r \, dr d\theta.$$

Example 6 *Evaluate the double integral of* $f(x, y) = e^{-(x^2+y^2)}$ *over the disk of radius* R *by changing to polar coordinates. Use the result to show that*

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$

Solution Let D be the disk of radius R. In polar coordinates the function f becomes $f(r\cos\theta, r\sin\theta) = e^{-r^2}$. Then,

$$\begin{aligned}
\iint_D e^{-(x^2+y^2)} \, dx \, dy &= \int_0^{2\pi} \int_0^R e^{-r^2} r \, dr d\theta \\
&= \int_0^{2\pi} \left(\int_0^R e^{-r^2} r \, dr \right) d\theta \\
&= \int_0^{2\pi} \left(-\frac{1}{2} e^{-r^2} \right) \Big|_0^R \, d\theta \\
&= \frac{1}{2} \left(1 - e^{-R^2} \right) \int_0^{2\pi} d\theta \\
&= \pi \left(1 - e^{-R^2} \right)
\end{aligned}$$

When $R \to \infty$ the disk D becomes the entire xy plane and

$$\iint_D e^{-(x^2+y^2)} \, dx \, dy = \pi.$$

Since we also have

$$\begin{aligned}
\iint_D e^{-(x^2+y^2)} \, dx \, dy &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} \, dx dy \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2
\end{aligned}$$

we get that

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$$

This is an informal evaluation of this important integral not possible to evaluate using only one variable calculus.

1.5 Change of Variables in Double Integrals

The polar coordinate change can be generalized to other change of variables. Suppose that a region Γ in the (u, v) plane is transformed into a region D in the xy-plane by the equations

$$x = x(u, v) \text{ and } y = y(u, v)$$

where and x and y are functions with continuous partial derivatives. The Jacobian of this transformation is the determinant

$$\begin{aligned} J(u, v) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \end{aligned}$$

Then, if f is continuous over D the following formula holds

$$\iint_D f(x, y)\, dx\, dy = \iint_\Gamma f(u, v)\ |J(u, v)|\ du\, dv.$$

For polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Then the Jacobian becomes

$$\begin{aligned} J(r, \theta) &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r. \end{aligned}$$

and the above formula reduces to

$$\iint_D f(x, y)\, dx\, dy = \iint_\Gamma f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

Example 7 *Let D be the plane region bounded by the lines*

$$2y - x = 0,\ \ y - 2x = 0,\ \ 2y - x = 2,\ \ y - 2x = 1,$$

and $f(x, y) = x + y$. Evaluate the integral

$$\iint_D f(x, y)\, dx\, dy$$

Solution Let's define the new variables u and v by

$$\begin{aligned} u &= 2y - x \\ v &= y - 2x \end{aligned}$$

Solving for x and y we get

$$\begin{aligned} x &= \frac{1}{3}(u - 2v) \\ y &= \frac{1}{3}(2u - v) \end{aligned}$$

The transformation defined by these equations maps the square $\Gamma : 0 \le u \le 2$; $0 \le v \le 1$ in the uv-plane into the region D in the xy-plane. The Jacobian is

$$\begin{aligned} J(u,v) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} \\ &= \frac{1}{3}. \end{aligned}$$

In addition

$$f(x,y) = x + y = u - v$$

Therefore,

$$\begin{aligned} \iint_D (x+y)\,dx\,dy &= \iint_\Gamma (u-v)\frac{1}{3}\,du\,dv \\ &= \int_0^1 \int_0^2 (u-v)\frac{1}{3}\,du\,dv \\ &= \frac{1}{3}\int_0^1 \left(\frac{u^2}{2} - vu\right)\Big|_0^2 dv \\ &= \frac{1}{3}\int_0^1 (2-2v)\,dv \\ &= \frac{1}{3}\left(2v - v^2\right)\Big|_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

2 Triple Integrals

2.1 Triple Integrals in Rectangular Coordinates

Let $f(x,y,z)$ be a function defined on the rectangular box or parallelepiped $R : a_1 \le x \le b_1$, $a_2 \le y \le b_2$, $a_3 \le z \le b_3$. In order to define the integral of f

over R we begin by dividing the intervals $[a_1, b_1]$, $[a_2, b_2]$ and $[a_3, b_3]$ into n, m and p subintervals

$$\begin{aligned} a_1 &= x_0 < x_1 < \cdots < x_n = b_1 \\ a_2 &= y_0 < y_1 < \cdots < y_m = b_2 \\ a_3 &= z_0 < z_1 < \cdots < z_m = b_3 \end{aligned}$$

This results in a partition Π of the rectangular box R into $n \times m \times p$ smaller cells

$$R_{ijk} : x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j,\ z_{k-1} \leq z \leq z_k$$

of volume

$$\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k = (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}).$$

The longest diagonal of all the cells R_{ijk}, i.e. the number

$$\|\Pi\| = \max\left(\sqrt{(\Delta x_i)^2 + (\Delta y_j)^2 + (\Delta z_k)^2}\right)$$

is called the norm of the partition.

Now we arbitrarily choose a sample point (ξ_i, η_j, χ_k) in each cell R_{ijk} and form the sum

$$\sum_{i,j,k} f(\xi_i, \eta_j, \chi_k)\, \Delta V_{ijk}.$$

If the function f is continuous and we refine the partition so that its norm $\|\Pi\|$ goes to zero, the sums approach a number independently of the choice of the points (ξ_i, η_j, χ_k). This number is called the triple integral of the function f over R and is denoted by

$$\iiint_R f(x, y, z)\, dV = \lim_{\|\Pi\| \to 0} \sum_{i,j,k} f(\xi_i, \eta_j, \chi_k)\, \Delta V_{ijk}.$$

Now, in the same way we did for double integrals, we can extend triple integrals to more general regions. Let D be a closed bounded region in space and R a rectangular box such that $D \subset R$.If f is a function defined on D we define an extension $f^{\ *}$ to the whole rectangle R by

$$f^*(x, y, z) = \begin{cases} f(x, y, z) & \text{if } (x, y, z) \in D \\ 0 & \text{if } (x, y, z) \in R - D \end{cases}.$$

If f is continuous on D the triple integral of f^* over R exits even though f^* might not be continuous at the boundary of D. Then we define the triple integral of f over the bounded region D by

$$\begin{aligned} \iiint_D f(x, y, z)\, dx\, dy\, dz &= \iiint_R f^*(x, y, z)\, dx\, dy\, dz \\ &= \lim_{\|\Pi\| \to 0} \sum_{i,j,k} f^*(\xi_i, \eta_j, \chi_k)\, \Delta V_{ijk}. \end{aligned}$$

If $f(x,y,z)=1$ the triple integral gives the volume of the region D

$$V=\iiint_D dx\,dy\,dz$$

As double integrals, triple integrals can be calculated by a three dimensional extension of the Fubini's Theorem. For instance if D is the region defined by

$$a\le x\le b,\ \varphi_1(x)\le y\le\varphi_2(x),\ \psi_1(x,y)\le z\le\psi_2(x,y)$$

we have

$$\iiint_D f(x,y,z)\,dx\,dy\,dz=\int_a^b\int_{\varphi_1(x)}^{\varphi_2(x)}\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Other iterated integrals can be formulated depending on the shape of the region D.

Example 8 *Find the volume enclosed by the intersection of the two cylinders* $x^2+y^2=1$ *and* $x^2+z^2=1$

Solution The projection of this region onto the plane xy is the circle $x^2+y^2=1$ or the plane region $-1\le x\le 1$, $-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$. The region is bounded above by the surface $z=\sqrt{1-x^2}$ and below by $z=-\sqrt{1-x^2}$. Using the symmetries of this region we integrate over the first octant and multiple by 8 to obtain

$$\begin{aligned}
V &= 8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} 1\,dz\,dy\,dx\\
&= 8\int_0^1\int_0^{\sqrt{1-x^2}} (z)\Big|_0^{\sqrt{1-x^2}}\,dy\,dx\\
&= 8\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx\\
&= 8\int_0^1 \left(y\sqrt{1-x^2}\right)\Big|_0^{\sqrt{1-x^2}}\,dx\\
&= 8\int_0^1 \left(1-x^2\right)\,dx\\
&= 8\left(x-\frac{1}{3}x^3\right)\Big|_0^1\\
&= \frac{16}{3}.
\end{aligned}$$

2.2 Triple Integrals in Cylindrical Coordinates

The cylindrical coordinates of a point (x, y, z) in space is the triple (r, θ, z) where r and θ are the polar coordinates of the vertical projection (x, y) onto the xy plane, with the limitation $r \geq 0$, $0 \leq \theta < 2\pi$, and z is the vertical rectangular coordinate. The following equations

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}
$$

and their inverse relations relations

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \arctan\frac{y}{x} \\
z &= z
\end{aligned}
$$

can be used to transform cylindrical coordinates into rectangular coordinates and vice versa.

In cylindrical coordinates surfaces $r = a$ are cylinders around the z axis of radius a; surfaces $\theta = \theta_0$ are vertical half planes containing the z axis and making an angle θ_0 with the x axis and surfaces $z = c$ are planes parallel to the xy plane.

Let $f(x, y, z)$ be a function defined on a region D in the xyz-space. When using cylindrical coordinates to evaluate the triple integral of f over D we can partition this region with cylindrical cells Ω_{ijk} made up of the points (r, θ, z) such that $r_{i-1} \leq r \leq r_i$, $\theta_{j-1} \leq \theta \leq \theta_j$ and $z_{k-1} \leq z \leq z_k$. The volume of these cells is given by

$$\Delta V_{ijk} = \frac{1}{2}\left(r_i + r_{i-1}\right)\Delta r_i \Delta\theta_j \Delta z_k$$

In the limit we get

$$dV = r\,dr\,d\theta\,dz$$

and the triple integral of f over D is given by

$$\iiint\limits_D f(x, y, z)\,dx\,dy\,dz = \iiint\limits_\Omega f(r\cos\theta, r\sin\theta, z) r\,dr\,d\theta\,dz$$

where Ω is the region D in $r\theta z$-space.

Example 9 *Evaluate the integral*

$$\iiint\limits_D \left(x^2 + y^2 + z^2\right)\,dx\,dy\,dz$$

where D is the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 2$ and $z = 0$.

Solution In cylindrical coordinates the region D is given by

$$0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 1, \ 0 \leq z \leq 2.$$

Therefore,

$$\begin{aligned}
\iiint_D \left(x^2 + y^2 + z^2\right)\big) \, dx \, dy \, dz &= \int_0^{2\pi} \int_0^1 \int_0^2 \left(r^2 + z^2\right) r \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 \int_0^2 \left(r^3 + z^2 r\right) \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 \left(r^3 z + \frac{z^3}{3} r\right)\Big|_0^2 \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 \left(2r^3 + \frac{8}{3} r\right) \, dr \, d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{2} r^4 + \frac{4}{3} r^2\right)\Big|_0^1 \, d\theta \\
&= \int_0^{2\pi} \frac{11}{6} \, d\theta \\
&= \frac{11}{3}\pi.
\end{aligned}$$

2.3 Triple Integrals in Spherical Coordinates

Given a point P in space its spherical coordinates is the triple (r, θ, φ) where:

1. r is the distance OP from the origin O to P, $r > 0$ for all points P different from the origin and $r = 0$ if P coincides with the origin
2. φ is the angle from the positive z-axis to the line OP and it is restricted to the interval $0 \leq \varphi \leq \pi$.
3. θ is the angle from the positive x axis to the line OQ where Q is the projection of P into the xy-plane. θ is restricted to the interval $0 \leq \theta \leq 2\pi$.

The following equations

$$\begin{aligned}
x &= r \sin\varphi \cos\theta \\
y &= r \sin\varphi \sin\theta \\
z &= r \cos\varphi
\end{aligned}$$

and their inverse relations relations

$$\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \arctan \frac{y}{x} \\
z &= \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}
\end{aligned}$$

can be used to transform spherical coordinates into rectangular coordinates and vice versa.

In spherical coordinates surfaces $r = a$ are spheres of radius a centered at the origin and surfaces $\theta = \theta_0$ are vertical half planes containing the z axis and making an angle θ_0 with the x axis.

Surfaces $\varphi = \varphi_0$ are the nappe of a cone generated by revolving around the z-axis the line emanating from the origin and forming an angle φ_0 with the positive z-axis. The surface $\varphi = 0$ reduces to the positive z-axis, the surface $\varphi = \pi/2$ to the plane $z = 0$ and the surface $\varphi = \pi$ to the negative z-axis.

Let $f(x, y, z)$ be a function defined on a region D in the xyz-space. When using spherical coordinates to evaluate the triple integral of f over D we can partition this region with spherical cells Ω_{ijk} made up of the points (r, φ, θ) such that $r_{i-1} \le r \le r_i$, $\varphi_{j-1} \le \varphi \le \varphi_j$ and $\theta_{k-1} \le \theta \le \theta_k$. For small increments the volume of these cells is given approximately by

$$\begin{aligned} \Delta V_{ijk} &= (r \sin\varphi\, \Delta\theta_k)(r\Delta\varphi_j)\Delta r_i \\ &= r^2 \sin\varphi\, \Delta r_i \Delta\varphi_j \Delta\theta_k \end{aligned}$$

In the limit we get

$$dV = r^2 \sin\varphi\, dr\, d\theta\, d\varphi$$

The triple integral of a function $f(x, y, z)$ over D is given by

$$\iiint\limits_D f(x, y, z)\, dx\, dy\, dz = \iiint\limits_\Omega F(r, \varphi, \theta)\, r^2 \sin\varphi\, dr\, d\varphi\, d\theta$$

where Ω is the region D in $r\varphi\theta$-space and

$$F(r, \varphi, \theta) = f(r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r \cos\varphi).$$

Example 10 *Find the volume of the solid enclosed by the spheres $r = 1$ and $r = 2\cos\varphi$*

Solution The sphere $r = 1$ is obviously the sphere of radius one centered at the origin

$$x^2 + y^2 + z^2 = 1.$$

The second one is the sphere of radius one centered at the point $(0, 0, 1)$. To see this multiply both sides of the equation $r = 2\cos\varphi$ by r to get

$$r^2 = 2r\cos\varphi.$$

Then using the transformation equations we obtain

$$x^2 + y^2 + z^2 = 2z$$

which is equivalent to

$$x^2 + y^2 + (z - 1)^2 = 1.$$

Let D be the region between these two spheres in rectangular coordinates and Ω the same region in spherical coordinates. At their intersection the following identity holds $1 = 2\cos\varphi$ and this implies $\varphi = \pi/3$. Therefore Ω is given by

$$0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{3},\ 0 \le r \le 1.$$

and the volume enclosed by the two spheres can be calculated as follows

$$\begin{aligned}
V &= \iiint_D dx\,dy\,dz \\
&= \iiint_\Omega r^2 \sin\varphi\, dr\, d\varphi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2 \sin\varphi\, dr\, d\varphi\, d\theta + \\
&\quad + \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2\cos\varphi} r^2 \sin\varphi\, dr\, d\varphi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}\sin\varphi\, d\varphi\, d\theta + \\
&\quad + \int_0^{2\pi}\int_{\pi/3}^{\pi/2} \frac{8}{3}\cos^3\varphi \sin\varphi\, d\varphi\, d\theta \\
&= \int_0^{2\pi} \frac{1}{3}\left(-\cos\varphi\right)\Big|_0^{\pi/3} d\theta + \\
&\quad + \int_0^{2\pi} \frac{2}{3}\left(-\cos^4\varphi\right)\Big|_{\pi/3}^{\pi/2} d\theta \\
&= \frac{1}{6}\int_0^{2\pi} d\theta + \frac{1}{24}\int_0^{2\pi} d\theta \\
&= \frac{5}{24}\int_0^{2\pi} d\theta \\
&= \frac{5\pi}{12}.
\end{aligned}$$

2.4 Change of Variables in Triple Integrals

The cylindrical and spherical coordinate changes can be generalized to other change of variables. Suppose that a region Ω in the (u, v, w) space is transformed into a region D in the xyz-space by the equations

$$x = x(u, v, w),\ y = y(u, v, w) \text{ and } z = z(u, v, w)$$

where and x, y and z are functions with continuous partial derivatives. The determinant

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

is called the Jacobian of the transformation. Then if f is continuous over D the following formula holds

$$\iiint\limits_D f(x,y,z)\,dx\,dy\,dz = \iiint\limits_\Omega F(u,v,w)\ |J(u,v,w)|\ du\,dv\,dw.$$

where

$$F(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w)).$$

In particular, for cylindrical coordinates

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

and the Jacobian is

$$\begin{aligned} J(r,\theta,z) &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r \end{aligned}$$

Therefore we obtain again

$$\iiint\limits_D f(x,y)\,dx\,dy\,dz = \iiint\limits_\Omega f(u,v,w) r\,dr\,d\theta\,dz.$$

For spherical coordinates

$$x = r\sin\varphi\cos\theta,\ y = r\sin\varphi\sin\theta,\ z = r\cos\varphi$$

and the Jacobian is

$$\begin{aligned} J(r,\varphi,\theta) &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{vmatrix} \\ &= \cos\varphi \begin{vmatrix} r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \end{vmatrix} \\ &\quad + r\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \cos\varphi\left[r^2\cos\varphi\sin\varphi\cos^2\theta + r^2\sin\varphi\cos\varphi\sin^2\theta\right] \\
&\quad + r\sin\varphi\left[r\sin^2\varphi\cos^2\theta + r\sin^2\varphi\sin^2\theta\right] \\
&= r^2\cos^2\varphi\sin\varphi + r^2\sin^3\varphi \\
&= r^2\sin\varphi
\end{aligned}
$$

Therefore

$$\iint\limits_{D} f(x,y,z)\,dx\,dy\,dz = \iint\limits_{\Omega} F(r,\varphi,z)r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

where

$$F(r,\varphi,\theta) = f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi).$$