CALCULUS

Midterm Exam
April 3, 2018
Duration: 45m

Exercise 1 *Find the values of x for which the geometric series*

$$\sum_{n=0}^{\infty}(-2)^{n+2}x^n = 4 - 8x + 16x^2 - 32x^3 + \cdots$$

converges. What's the sum of the series for those values of x?

Solution Factoring out 4 we get

$$\begin{aligned}\sum_{n=0}^{\infty}(-2)^{n+2}x^n &= 4\sum_{n=0}^{\infty}(-2)^n x^n \\ &= 4\sum_{n=0}^{\infty}(-2x)^n\end{aligned}$$

This is geometric series of ratio $r = -2x$. A geometric series of ratio r is convergent if and only if $|r| < 1$ and in this case its sum is

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Therefore the given series is convergent if and only if $|-2x| < 1$, i.e. when

$$-\frac{1}{2} < x < \frac{1}{2}.$$

and its sum is

$$\begin{aligned}\sum_{n=0}^{\infty}(-2)^{n+2}x^n &= 4\sum_{n=0}^{\infty}(-2x)^n \\ &= \frac{4}{1+2x}.\end{aligned}$$

Exercise 2 *Find the maximum value of the function $f(x) = 3x - 2x^2$. Also, find the average value of f on the interval where the function is positive.*

Solution The graph of this function is a parabola. The x-intercepts are the solutions of the equation $f(x) = 0$. Since

$$f(x) = 3x - 2x^2 = x(3 - 2x)$$

the intercepts are the points $x = 0$ and $x = 3/2$.

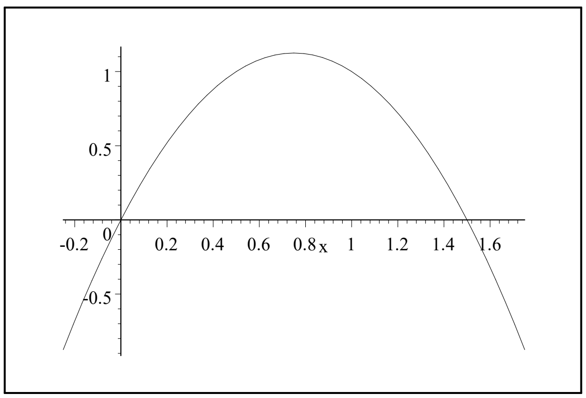
The critical points of f are the solutions of the equation $f'(x) = 0$.Since.

$$f'(x) = 3 - 4x$$

the only critical point is $x = 3/4$. At this point the parabola reaches its maximum value

$$V_{\max}(f) = f(3/4) = 9/8.$$

Plot of the graph of $f(x) = 3x - 2x^2$



The average value of f in the interval $[0, 3/2]$ is given by the integral

$$\begin{aligned} V_{avg}(f) &= \frac{1}{3/2 - 0} \int_0^{3/2} \left(3x - 2x^2\right) dx \\ &= \frac{2}{3} \left(3\frac{x^2}{2} - 2\frac{x^3}{3}\right)\Big|_0^{3/2} \\ &= \frac{3}{4}. \end{aligned}$$