CALCULUS

Midterm Exam
March 31, 2017
Duration: 45m

Exercise 1 *Find, if possible, the value of a such that*

$$1 - \frac{a}{2} + \frac{a^2}{4} - \frac{a^3}{8} + \cdots = S$$

if

1. $S = 1/4$

2. $S = 3/4$

Solution The series is a geometric series of ratio $r = -\frac{a}{2}$.The series converges if and only if $|r| < 1$, i.e. $|a| < 2$. In this case the sum is

$$S = \frac{1}{1-r} = \frac{1}{1+\frac{a}{2}} = \frac{2}{2+a}$$

and solving for a we obtain

$$a = \frac{2(1-S)}{S}.$$

1. If $S = 1/4$ then

$$a = \frac{2\left(1 - \frac{1}{4}\right)}{\frac{1}{4}} = 6$$

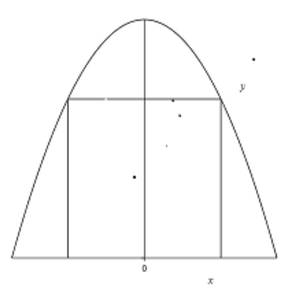
Since $|a| \geq 2$ the series is divergent. Therefore there is no value of a that makes the series convergent with sum $S = 1/4$.

2. If $S = 3/4$ then

$$a = \frac{2\left(1 - \frac{3}{4}\right)}{\frac{3}{4}} = \frac{2}{3}.$$

Now the ratio of the series is $r = -1/3$ and the series converges. Therefore $a = 2/3$ yields a convergent series with sum $S = 3/4$.

Exercise 2 *Find the dimensions of the rectangle with largest area that has its base on the x-axis and its upper two vertices on the parabola $y = a^2 - x^2$ with $a > 0$.*



Solution

1. The area of the rectangle is given by

$$\begin{aligned} A &= 2xy \\ &= 2x\left(a^2 - x^2\right) \\ &= 2a^2x - 2x^3 \end{aligned}$$

with $0 \leq x \leq a$.The derivative of this function is

$$A' = 2a^2 - 6x^2.$$

Solving $A' = 0$ for x yields

$$x^2 = \frac{1}{3}a^2.$$

and, therefore, there is only one critical point in the interval $(0, a)$

$$x_0 = \frac{\sqrt{3}}{3}a$$

Since $A(0) = A(a) = 0$ and

$$A(x_0) = 2\frac{\sqrt{3}}{3}a\left(a^2 - \frac{1}{3}a^2\right) = \frac{4\sqrt{3}}{9}a^3 > 0$$

the critical point corresponds to an absolute maximum. Therefore the dimensions of the rectangle of largest area are:

$$\begin{aligned} base &= \frac{2\sqrt{3}}{3}a \\ height &= \frac{2}{3}a^2. \end{aligned}$$