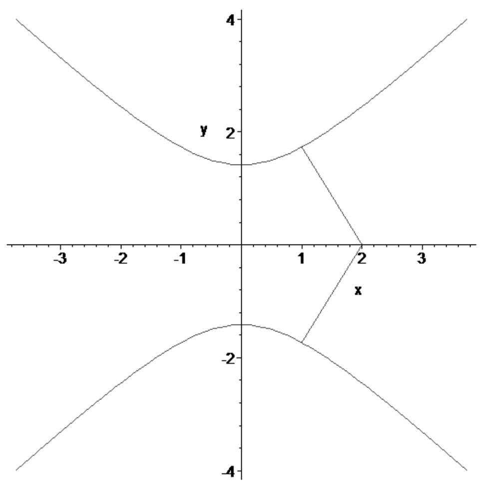
CALCULUS

Midterm Exam
April 1, 2016
Duration: 45m

Exercise 1 *(7p) Find the point on the curve $y^2 - x^2 = 2$ closest to the point $P(1,0)$ and the shortest distance from P to the curve.*

Solution Plot of the curve $y^2 - x^2 = 2$



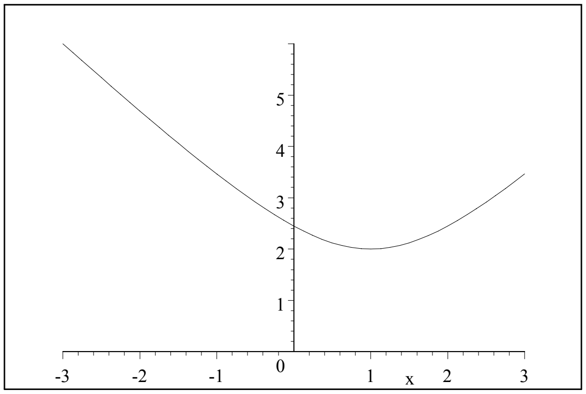
The distance d from the point P to a generic point (x, y) on the curve is given by

$$\begin{aligned} d &= \sqrt{(x-2)^2 + (y-0)^2} \\ &= \sqrt{(x^2 - 4x + 4) + y^2} \end{aligned}$$

From the equation of the curve we get $y^2 = 2 + x^2$. Using this relationship yields

$$\begin{aligned} d &= \sqrt{(x^2 - 4x + 4) + (2 + x^2)} \\ &= \sqrt{2x^2 - 4x + 6} \end{aligned}$$

Plot of d as a function of x



Since $(d^2)' = 2dd'$ and $d \neq 0$,finding the points where $d' = 0$ is equivalent to finding the points where the derivative of

$$d^2 = 2x^2 - 4x + 6$$

equals zero, i.e. the solution(s) of the equation

$$4x - 4 = 0.$$

The only solution is $x = 1$. Therefore $x = 1$, $y = \pm\sqrt{3}$ are the points on the curve closest to the point $(2, 0)$ and the minimum distance to the curve is

$$d_{\min} = 2.$$

Exercise 2 *(6p) Find the average (mean) value of the function $y = \sqrt{x}$ on the interval $[0, 2]$. Graph the function and explain the meaning of this average value. Does the function take on its average value at some point in the given interval?*

Solution The average value of a function $y = f(x)$ on the interval $[a, b]$ is defined by the integral

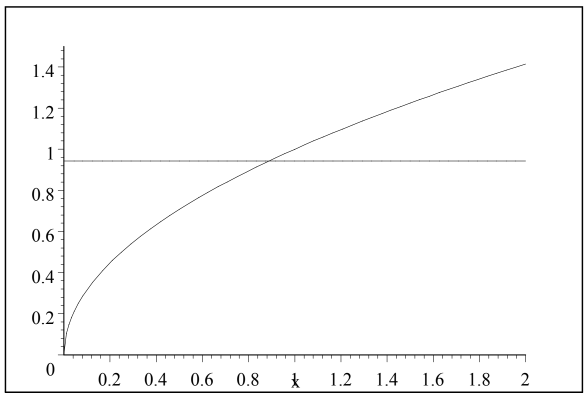
$$\bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

In this case

$$\bar{y} = \frac{1}{2}\int_0^2 \sqrt{x}\,dx$$

$$
\begin{aligned}
&= \frac{1}{2} \frac{x^{3/2}}{\frac{3}{2}}\Bigg|_0^2 \\
&= \frac{2\sqrt{2}}{3}.
\end{aligned}
$$

The average value is the height of a rectangle whose area is the same as the are under the graph of the function



The average value is taken at the point x such that

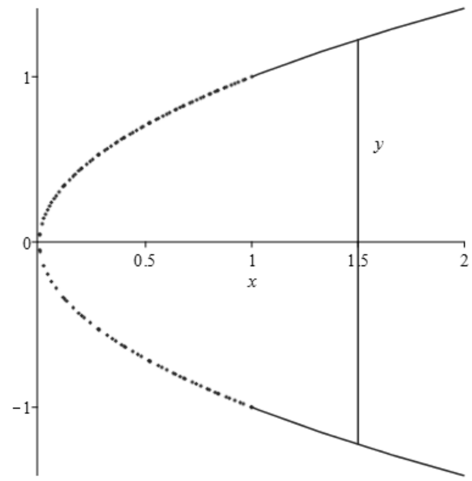
$$\sqrt{x} = \frac{2\sqrt{2}}{3}$$

i.e. at the point

$$x = \frac{8}{9}.$$

Exercise 3 *(7p) The base of a solid is the region bounded by the parabola* $y^2 = x$ *and the lines* $x = 1$ *and* $x = 2$ *Find the volume of the solid if its cross sections perpendicular to the* x*-axis are isosceles right triangles having the hypotenuse in the plane of the base.*

Solution Plot of the parabola $y^2 = x$ and the base of the cross section at distance x from the origin



The cross sections are isosceles right triangles with hypotenuse $2y$. Let l be the legs of the triangles, by the Pythagoras's theorem we have $2l^2 = 4y^2$. Then the area of a cross section is given by

$$A(x) = \frac{1}{2}l^2 = y^2 = x$$

and the volume of the solid is

$$\begin{aligned} V &= \int_1^2 A(x)\,dx \\ &= \int_1^2 x\,dx \\ &= \left.\frac{x^2}{2}\right|_1^2 \\ &= \frac{3}{2} \end{aligned}$$