CALCULUS

Midterm Exam
March 26, 2014
Duration: 45m

Exercise 1 *Find, if possible, the value of a such that*

$$1 - e^a + e^{2a} - e^{3a} + \cdots = S$$

if

1. $S = 1/4$
2. $S = 4/5$

Solution The series is a geometric series of ratio $r = -e^a$.If the sries converges the sum is

$$S = \frac{1}{1-r} = \frac{1}{1+e^a}.$$

1. If we want $S = 1/4$ then a needs to be a solution of the equation

   $$\frac{1}{1+e^a} = \frac{1}{4}.$$

   This implies

   $$1 + e^a = 4 \qquad \text{and} \qquad e^a = 3.$$

   In this case the ratio of the series should be $r = -3$ and the series would diverge. Therefore there is no value of a that makes the series convergent with sum $S = 1/4$.

2. If $S = 4/5$ then a needs to be a solution of the equation

   $$\frac{1}{1+e^a} = \frac{4}{5}.$$

   This implies

   $$1 + e^a = \frac{5}{4} \qquad \text{and} \qquad e^a = \frac{1}{4}.$$

   Now the ratio of the series should be $r = -1/4$ and the series would converge. Therefore $a = -\ln 4$ yields a convergent series with sum $S = 4/5$.

Exercise 2 *A 1 meter long wire is cut into two pieces. One piece is bent into a circle and the other one into a square. Where should the wire be cut to maximize the total area enclosed by both figures.*

Solution Let x be the the length of the piece of wire bent into a circle. Then $1-x$ is the length of the piece bent into a square. The radius of the resulting circle will be $r = x/2\pi$ and the side of the square will $l = (1-x)/4$. Therefore the total area enclosed by the two shapes is

$$\begin{aligned} A &= \pi r^2 + l^2 \\ &= \pi \left(\frac{x}{2\pi}\right)^2 + \left(\frac{1-x}{4}\right)^2 \\ &= \frac{x^2}{4\pi} + \frac{(1-x)^2}{16}. \end{aligned}$$

The derivative of this function is

$$\begin{aligned} A' &= \frac{x}{2\pi} - \frac{1-x}{8} \\ &= \frac{(4+\pi)x - \pi}{8\pi}. \end{aligned}$$

The condition $A' = 0$ produces a critical point

$$x = \frac{\pi}{4+\pi}.$$

Since

$$A'' = \frac{4+\pi}{8} > 0$$

the second derivative shows that the critical point corresponds to a minimum. Comparing the values of A at the endpoints we have

$$\begin{aligned} A(0) &= \frac{1}{16} \\ A(1) &= \frac{1}{4\pi}. \end{aligned}$$

Therefore to maximize the area choose $x = 1$, i.e. use the whole wire to form a circle. Plot of the function between $x = 0$ and $x = 1$