Series

1 Infinite Series

Let $a_0, a_1, \ldots, a_n, \ldots$ be a sequence of infinite numbers. An **infinite series** is an expression of the form

$$a_0 + a_1 + \cdots + a_n + \cdots$$

which is represented in compact form by the symbol

$$\sum_{n=0}^{\infty} a_n$$

or even $\sum a_n$ when summation from 0 to ∞ is understood. The number a_n is called the n**th term** of the series. To define the sum of an infinite series we start building the sequence

$$\begin{aligned} S_0 &= a_0 \\ S_1 &= S_0 + a_1 = a_0 + a_1 \\ &\vdots \\ S_n &= S_{n-1} + a_n = a_0 + \cdots + a_n = \sum_{i=0}^{n} a_i \\ &\vdots \end{aligned}$$

This sequence is called the **sequence of partial sums** of the series and the number S_n the n**th partial sum**. If the sequence of partial sums converges to a finite limit S we say that the series **converges** and that its **sum** is S. In this case we write

$$\sum_{n=0}^{\infty} a_n = \lim_{n \to +\infty} S_n = S.$$

If the sequence of partial sums converges to infinity or does not converge we say that the series **diverges**.

When a series is convergent the difference between its sum S and a partial sum S_n

$$R_n = S - S_n = \sum_{k=n+1}^{\infty} a_k$$

is called the remainder of order N and it represents the error made in approximating the sum S of a series with the nth partial sum S_n.

Convergent series can be added up or multiplied by a constant. Given two convergent series $\sum a_n = A$ and $\sum b_n = B$ and a constant λ we have

$$\begin{aligned}\sum_{n=0}^{\infty}(a_n+b_n) &= A+B\\ \sum_{n=0}^{\infty}\lambda a_n &= \lambda A.\end{aligned}$$

We note that the sum of a convergent series is the limit of the partial sequence S_n and not the limit of the term sequence a_n. Actually, if a series $\sum a_n$ is convergent, then

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty}(S_n - S_{n-1}) = 0$$

what is a necessary condition for convergence.

Geometric Series A geometric series is a series of the form

$$1 + r + r^2 + \cdots + r^n + \cdots$$

where r is a fixed real number called **ratio**.

If $r = 1$,an expression for the nth partial sum is straightforward

$$S_n = \overbrace{1+\cdots+1}^{n \text{ times}} = n$$

therefore

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} n = +\infty$$

and the series diverges.

If $r \neq 1$ an expression for the nth partial sum can be obtained the following way. Start with

$$S_n = 1 + r + r^2 + \cdots + r^n$$

multiply both sides by r

$$rS_n = r + r^2 + \cdots + r^n + r^{n+1}$$

subtract rS_n from S_n

$$(1-r)\,S_n = 1 - r^{n+1}$$

and solve for S_n

$$S_n = \frac{1-r^{n+1}}{1-r}.$$

If $|r| < 1$ then $r^{n+1} \to 0$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\frac{1-r^{n+1}}{1-r} = \frac{1}{1-r}$$

so the series converges.

If $r > 1$, then $r^{n+1} \to +\infty$ as $n \to +\infty$ and $\lim_{n\to+\infty} S_n = +\infty$. If $r < -1$, the sequence S_n does not have a limit because it alternates signs and $|r|^{n+1} \to +\infty$ as $n \to +\infty$. Last, if $r = -1$, the sequence S_n alternates between 1 or 0 and does not have a limit. In all these cases the series diverges.

Conclusion 1 *If $|r| < 1$ the geometric series*

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots$$

converges and its sum is

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

If $|r| \geq 1$ the geometric series diverges.

Example 2 *Find the sum of the series*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Solution It is a geometric series with ratio $r = 1/2$. Since $|r| < 1$ the series converges and its sum is

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2.$$

Example 3 *Find the sum of the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4}{3^n} = -\frac{4}{3} + \frac{4}{9} - \frac{4}{27} + \cdots$$

Solution Factoring out $-4/3$ we obtain a geometric series with ratio $r = -1/3$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4}{3^n} = -\frac{4}{3}\left(1 - \frac{1}{3} + \frac{1}{27} - \cdots\right) = -\frac{4}{3}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n$$

Since $|r| < 1$ the last series converges and

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4}{3^n} = -\frac{4}{3}\frac{1}{1-\left(-\frac{1}{3}\right)} = -1.$$

2 Convergence Tests for Series

For the geometric series we succeeded in finding an expression for the nth partial sum and computing its sum directly from it. Unfortunately, finding an expression for the nth partial sum of a series is not possible in most cases. In the absence of these formulas we have to resort to establishing series convergence indirectly and then approximating the sum of the series by a partial sum. In this section we review the most usual tests used to establish the convergence of a series.

Criterion 4 (The nth-Term Test for Divergence) *If* $\lim_{n\to+\infty} a_n$ *does not exist or is different from zero the series* $\sum_{n=1}^{+\infty} a_n$ *diverges.*

Example 5 *Show that the following series are divergent*

$$\textit{1)} \sum_{n=0}^{+\infty} n, \ \textit{2)} \sum_{n=0}^{+\infty} (-1)^n, \ \textit{3)} \sum_{n=1}^{+\infty} \frac{n-1}{n}.$$

Solution

1. Diverges because $\lim_{n\to+\infty} n = +\infty$
2. Diverges because $\lim_{n\to+\infty}(-1)^n$ does not exist
3. Diverges because $\lim_{n\to+\infty} \frac{n-1}{n} = 1 \neq 0$.

Criterion 6 (The Integral Test) *Let f be a continuous, positive, decreasing function for all $x > 1$ where N is a positive integer. Then the series* $\sum_{n=1}^{+\infty} a_n$*, where $a_n = f(n)$, is convergent if and only if the integral*

$$\int_1^{+\infty} f(x)\,dx$$

is convergent. In this case we have the following estimate of the remainder of the series

$$\int_{n+1}^{+\infty} f(x)\,dx \leq R_n \leq \int_n^{+\infty} f(x)\,dx.$$

Example 7 (The Harmonic Series) *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent.

Solution The function $f(x) = 1/x$ is continuous, positive and decreasing for $x \geq 1$. Since the integral

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x}\,dx &= \lim_{b\to+\infty} \int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to+\infty} (\ln b - \ln 1) \\ &= +\infty\end{aligned}$$

is divergent, the harmonic series is also divergent by the Integral Test.

Example 8 (The p-Series) *Given a real constant p, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

converges if $p > 1$ and diverges if $p \leq 1$..

Solution If $p > 1$ the function $f(x) = 1/x^p$ is continuous, positive and decreasing for $x \geq 1$. Since

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x^p}\,dx &= \lim_{b\to+\infty} \int_1^b \frac{1}{x^p}\,dx \\ &= \frac{1}{-p+1} \lim_{b\to+\infty} \left(b^{-p+1} - 1\right) \\ &= \frac{1}{1-p}(0-1) \\ &= \frac{1}{p-1}\end{aligned}$$

the series is convergent by the Integral Test.

If $p = 1$ we have the harmonic series which is divergent (see above example).

If $0 < p < 1$ we can still apply the Integral Test and since

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x^p}\,dx &= \lim_{b\to+\infty} \int_1^b \frac{1}{x^p}\,dx \\ &= \frac{1}{-p+1} \lim_{b\to+\infty} \left(b^{-p+1} - 1\right) \\ &= +\infty\end{aligned}$$

the series diverges.

If $p \leq 0$

$$\lim_{n\to+\infty} \frac{1}{n^p} = \lim_{n\to+\infty} n^{-p} = \begin{cases} +\infty \text{ if } p < 0 \\ 1 \text{ if } p = 0 \end{cases}$$

so the series diverges by the nth-Term Test for Divergence.

Criterion 9 (Direct Comparison Test) *Let $\sum a_n$ and $\sum b_n$ two series such that $0 < a_n \leq b_n$.*

1. *If $\sum b_n$ converges, the $\sum a_n$ converges.*
2. *If $\sum a_n$ diverges, the $\sum b_n$ diverges.*

Criterion 10 (The Limit Comparison Test) *Let $\sum a_n$ and $\sum b_n$ two series such that $0 < a_n$ and $0 < b_n$ and*

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = L.$$

1. *If $L > 0$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.*
2. *If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.*
3. *If $L = +\infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

Example 11 *Show convergence of the series*

$$\sum_{n=1}^{\infty} \frac{n+1}{n^3+2n+4}.$$

Solution Let a_n be the general term of the given series. For large n we expect a_n to behave like $n/n^3 = 1/n^2$. Then, we set $b_n = 1/n^2$. The series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p-series with $p > 1$, so it converges. Now we compute the limit

$$\begin{aligned} L &= \lim_{n\to+\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to+\infty} \frac{\frac{n+1}{n^3+2n+4}}{\frac{1}{n^2}} \\ &= \lim_{n\to+\infty} \frac{n^3+n^2}{n^3+2n+4} \\ &= \lim_{n\to+\infty} \frac{1+1/n}{1+2/n^2+4/n^3} \\ &= 1. \end{aligned}$$

Since $L > 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

Criterion 12 (The Alternating Series Test) *Let a_n, $n \geq 1$, be a sequence such that*

1. *$a_n \geq 0$ for all n.*

2. $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$

3. $lim_{n\to+\infty} a_n = 0$.
   Then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

converges and

$$|R_n| \leq a_{n+1}.$$

Example 13 *The alternating harmonic series*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent and

$$|R_n| \leq \frac{1}{n+1}.$$

Solution The sequence $a_n = 1/n$ satisfies the three requirements of the Alternating Test so the alternating harmonic series is convergent and the error estimate applies.

Criterion 14 (The Absolute Ratio Test) *Let* $\sum a_n$ *be a series and*

$$L = \lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|}.$$

1. *If* $L < 1$ *the series converges.*

2. *If* $L > 1$ *the series diverges.*

3. *If* $L = 1$ *the test is inconclusive.*

Example 15 *The series*

$$\sum_{n=1}^{\infty}\frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$$

is convergent.

Solution For this series

$$\begin{aligned} L &= \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} \\ &= \lim_{n\to+\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \\ &= \lim_{n\to+\infty} \frac{n!}{(n+1)!} \\ &= \lim_{n\to+\infty} \frac{1}{n+1} \\ &= 0. \end{aligned}$$

Since $L = 0 < 1$ the series converges by the ratio test.

Criterion 16 (The Absolute Root Test) *Let $\sum a_n$ be a series and*

$$L = \lim_{n\to+\infty} \sqrt[n]{|a_n|}.$$

1. *If $L < 1$ the series converges.*
2. *If $L > 1$ the series diverges.*
3. *If $L = 1$ the test is inconclusive.*

Example 17 *The series*

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$$

is convergent.

Solution For this series

$$\begin{aligned} L &= \lim_{n\to+\infty} \sqrt[n]{|a_n|} \\ &= \lim_{n\to+\infty} \sqrt[n]{\left|\frac{(-2)^n}{n^n}\right|} \\ &= \lim_{n\to+\infty} \sqrt[n]{\frac{2^n}{n^n}} \\ &= \lim_{n\to+\infty} \frac{2}{n} \\ &= 0. \end{aligned}$$

Since $L = 0 < 1$ the series converges by the root test.

3 Power Series

Power series are infinite polynomials. A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

or more generally

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots .$$

in which $a_0, a_1, \ldots, a_n, \ldots$ are constants called **coefficients** and c another constant called **center**.

Convergence of a power series depends on x and the sum will be a function of x.Nevertheless applying the ratio test it is possible to show that there exists a positive number R called the **radius of convergence** such that

1. the series converges for $|x| < R$ (or $|x-c| < R$),
2. diverges for $|x| > R$ (or $|x-c| > R$) and
3. may or may not converge at the endpoints $x = \pm R$ (or $x = c \pm R$).

The radius can also be zero or infinity. If $R = 0$, the series only converges at the center. If $R = \infty$, the series converges for all x.

Example 18 *The geometric series* $\sum_{n=0}^{\infty} x^n$ *converges if and only* $|x| < 1$ *and*

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots , \qquad (-1 < x < 1) .$$

Within the interval of convergence, power series can be manipulated the same way as polynomials, we can add, multiply or divide them. We can also make substitutions, differentiate and integrate them.

Example 19 *Show that*

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots , \qquad (-1 < x < 1) .$$

Solution Substitute $-t$ for x in the geometric series to obtain

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots , \qquad (-1 < t < 1) .$$

Now integrate both sides

$$\int_0^x \frac{1}{1+t}\, dt = \int_0^x \left(1 - t + t^2 - t^3 + \cdots\right) dt$$

to obtain

$$\begin{aligned}\ln(1+x) &= \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right)\Big|_0^x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\end{aligned}$$

if and only if $-1 < x < 1$.

Example 20 *Show that*

$$\arctan x = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad (-1 < x < 1).$$

Solution Substitute $-t^2$ for x in the geometric series to obtain

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots, \qquad (-1 < t < 1).$$

Now integrate both sides

$$\int_0^x \frac{1}{1+t^2}\, dt = \int_0^x \left(1 - t^2 + t^4 - t^6 + \cdots\right)\, dt$$

to obtain

$$\begin{aligned}\arctan(1+x) &= \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots\right)\Big|_0^x \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\end{aligned}$$

if and only if $-1 < x < 1$.

Taylor and Maclaurin Series Let $f(x)$ be infinitely differentiable function. A **Taylor series generated by** f **at** $x = c$ is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots.$$

When $c = 0$ the series is known as the **Maclaurin series generated by** f

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

The nth partial sum of these series is called the **Taylor polynomial of order** n generated by f at $x = c$

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and the difference between the generating function $f(x)$ and its Taylor polynomial of order n is called the **remainder of order** n

$$R_n(x) = f(x) - P_n(x).$$

Given the Taylor series generated by a function f three important questions arise:

1. Where does this series converge?
2. Does it converge to its generating function?
3. If so, how accurate is the approximation of the function by its Taylor polynomials?

The answer to these questions can be obtained using the following formula of the remainder:

Theorem 21 (Lagrange's Formula for the Remainder) *If f has continuous derivatives up to order $n+1$ on an interval centered at $x = c$, for each x on that interval there exists a number ξ between c and x such that*

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}.$$

The number ξ is unknown but if we can find a constant M such that $\left|f^{(n+1)}(t)\right| \leq M$ for all t between c and x, then

$$|R_n(x)| \leq M\frac{(x-c)^{n+1}}{(n+1)!}.$$

In this case $R_n(x) \to 0$ as $n \to +\infty$. and we can conclude that the Taylor series converges to $f(x)$.

Example 22 *The Maclaurin series for e^x is*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad (-\infty < x < +\infty)$$

Solution If $f(x) = e^x$ first we compute all derivatives

$$f'(x) = e^x,\ f''(x) = e^x, \ldots,\ f^{(n)}(x) = e^x, \ldots$$

and then we evaluate these derivatives at $x = 0$

$$f(0) = 1,\ f'(0) = 1,\ f''(0) = 1, \ldots,\ f^{(n)}(0) = 1, \ldots.$$

The Maclaurin series generated by f is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Now we need to show that it converges to the generating function $f(x) = e^x$. For each t between 0 and x we have $\left|f^{(n+1)}(t)\right| = e^t \leq e^{|x|} = M$. Therefore $R_n(x) \to 0$ as $n \to +\infty$ and the series converges to e^x for all x.

Example 23 *The Maclaurin series for* $\sin x$ *is*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots, \qquad (-\infty < x < +\infty)$$

Solution If $f(x) = \sin x$ first we compute all derivatives

$$f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x,\ f^{(4)}(x) = \sin x, \ldots$$

Observe that the derivatives repeat every four terms, their values at $x = 0$ are

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{(4)}(0) = 0, \ldots$$

The Maclaurin series generated by f is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Now we need to show that it converges to the generating function $f(x) = \sin x$. For each t between 0 and x we have $\left|f^{(n+1)}(t)\right| \leq 1 = M$. Therefore $R_n(x) \to 0$ as $n \to +\infty$ and the series converges to $\sin x$ for all x.