Integrals

1 Initial Value Problems

A simple initial value problem (IVP): find a function $y(x)$ such that solves the differential equation

$$\frac{dy}{dx} = f(x)$$

and meets the initial condition

$$y(x_0) = y_0.$$

To solve this problem follow the following steps:

1. Find the general solution of the differential equation by integration. The general solution is given by

   $$y = F(x) + C$$

   where $F(x)$ is an antiderivative of $f(x)$.

2. Find the particular solution that satisfies the initial condition. The particular value of C is obtained from the equation

   $$y_0 = F(x_0) + C$$

   and the particular solution of the IVP is

   $$y = y_0 + F(x) - F(x_0).$$

Example 1 *Solve the IVP*

$$\frac{dy}{dx} = x^2; \qquad y(0) = -1;$$

Solution

1. Find the general solution of the differential equation by integration. The differential equation

   $$\frac{dy}{dx} = x^2$$

Integrate both sides of the equation

$$\int \frac{dy}{dx}\,dx = \int x^2\,dx$$

Evaluate the integrals

$$y + C_1 = \frac{x^3}{3} + C_2$$

Combine the constants as one

$$y = \frac{x^3}{3} + C$$

2. Find the particular solution of the IVP. Apply the initial condition to the general solution

$$\begin{aligned} y_0 &= \frac{x_0^3}{3} + C \\ -1 &= \frac{0}{3} + C \end{aligned}$$

Solve for C

$$C = -1$$

The particular solution is

$$y = \frac{x^3}{3} - 1.$$

Variable separation method The differential equation involved in an initial value problem may include the unknown function $y(x)$ in the right hand side of the equation. In these cases the general solution will be harder to obtain and their study belong to a course on ordinary differential equations. Next examples show how to deal with some differential equations that can still be handled easily by the so called variable separation method. In these cases all terms related to the independent variable x can be left on the right hand side and all the terms related to the the dependent variable or unknown function y can be moved to the left hand side.

Example 2 *Find the curve that goes through the point* $(1, 3)$ *and whose slope at each point is* $-x/y$.

Solution This problem is equivalent to solve the following IVP

$$\frac{dy}{dx} = -\frac{x}{y}; \qquad y(1) = 3;$$

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

Start separating the variables in the following way

$$y\,dy = -x\,dx$$

Now integrate both sides

$$\int y\,dy = -\int x\,dx$$

Evaluate the integrals

$$\frac{y^2}{2} + C_1 = -\frac{x^2}{2} + C_2$$

Rearrange terms

$$x^2 + y^2 = 2(C_2 - C_1)$$

Call the constant $2(C_2 - C_1)$ just C to obtain the general solution

$$x^2 + y^2 = C$$

2. Apply the initial condition to the general solution to obtain

$$\begin{aligned} x_0^2 + y_0^2 &= C \\ 1^2 + 3^2 &= C \\ C &= 10 \end{aligned}$$

The particular solution of the IVP is

$$x^2 + y^2 = 10.$$

Example 3 *Solve the IVP*

$$\frac{dy}{dx} = -ay; \qquad y(0) = y_0;$$

where a is a constant greater than 0.

Solution

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = -ay$$

Start separating the variables in the following way

$$\frac{dy}{y} = -a\,dx$$

Now integrate both sides

$$\int \frac{dy}{y} = -\int a\,dx$$

Evaluate the integrals

$$\ln|y| = -ax + C$$

Exponentiate

$$|y| = e^{-ax+C}$$

Apply the law of exponents

$$|y| = e^{C}e^{-ax}$$

Solve for y

$$y = \pm e^{C}e^{-ax}$$

Call the constant $\pm e^{C}$ just K

$$y = Ke^{-ax}$$

Since the function $y = 0$ is also a solution of the differential equation by allowing K to take on the value 0 as well the last formula represents the general solution of the differential equation.

2. Apply the initial condition to the general solution to obtain

$$y_0 = Ke^{-ax_0} = Ke^{-a0} = K.$$

The particular solution of the IVP is

$$y = y_0e^{-ax}.$$

2 Definite Integrals

This section is a quick review of the main results about definite integrals needed for applications. To define the definite integral of a continuous function $f(x)$ on an interval $[a, b]$ we divide the interval into n consecutive subintervals of the same length $\Delta x = (b - a)/n$ and select a point c_i in each subinterval.

Then we construct a vertical rectangle from each subinterval to the point $(c_i, f(c_i))$ on the graph of f and form the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x.$$

The definite integral of f from a to b is defined as the limit of the above sum as $n \to +\infty$

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

When f is positive each product $f(c_i)\Delta x_i$ represents the area of the vertical rectangle. In this case the sum represents an approximation of the area under the graph of f and the definite integral the exact area. When f is both positive and negative the definite integral defines a signed or net area under the graph.

The first part of the fundamental theorem of Calculus shows that integration is the reverse process of differentiation.

Theorem 4 (The Fundamental Theorem of Calculus Part I) *If f is a continuous function on $[a, b]$, then the function*

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of $f(x)$, i.e. $F(x)$ has a derivative at any point of $[a, b]$ and $F'(x) = f(x)$.

The second part of this theorem greatly simplifies the evaluation of definite integrals when an antiderivative of $f(x)$ is known.

Theorem 5 (The Fundamental Theorem of Calculus Part II) *If f is a continuous function on $[a, b]$, and $F(x)$ is an antiderivative of $f(x)$ then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

If an antiderivative is not known definite integrals have to be evaluated approximately by numerical procedures.

Example 6 *Evaluate the following integral*

$$\int_0^2 \left|1 - x^2\right|\,dx$$

Solution Since the function is positive between $x = 0$ and $x = 1$ and negative between $x = 1$ and $x = 2$ we start decomposing the integral

$$\begin{aligned}\int_0^2 \left|1 - x^2\right|\,dx &= \int_0^1 \left|1 - x^2\right|\,dx + \int_1^2 \left|1 - x^2\right|\,dx \\ &= \int_0^1 \left(1 - x^2\right)\,dx - \int_1^2 \left(1 - x^2\right)\,dx.\end{aligned}$$

An antiderivative of the function $1-x^2$ is $x-x^3/3$ therefore

$$\begin{aligned}\int_0^2 \left|1-x^2\right| dx &= \left(x-\frac{x^3}{3}\right)\Big|_0^1 - \left(x-\frac{x^3}{3}\right)\Big|_1^2 \\ &= \left[\left(1-\frac{1}{3}\right)-\left(1-\frac{0}{3}\right)\right]-\left[\left(1-\frac{8}{3}\right)-\left(1-\frac{1}{3}\right)\right] \\ &= 2.\end{aligned}$$

Average Value of a Function From basic arithmetic we know that the average of n numbers is their sum divided by n. If we consider a continuous function f on a closed interval $[a,b]$ there may be an infinite number of values to consider. In this case we can start by dividing the interval $[a,b]$ into n consecutive subintervals of the same length $\Delta x = (b-a)/n$ and evaluating the function f at a point c_i in each subinterval. The average value of these n sampled values gives an approximation of the average value of the function

$$\begin{aligned} AV(f) &\simeq \frac{f(c_1)+\cdots+f(c_n)}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n} f(c_i) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\,\Delta x. \end{aligned}$$

Letting $n \to +\infty$ leads to the following definition.

Definition 7 *The average or mean value of a function $f(x)$ on an interval $[a,b]$ is given by*

$$AV(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

For a positive function this number gives the height of a rectangle that encloses the same area as the graph of the function.

Example 8 *Find the average value of the function $f(x)=\sqrt{1-x^2}$ on the interval $[0,1]$.*

Solution The average value is given by

$$\begin{aligned} AV(f) &= \frac{1}{1-0}\int_0^1 \sqrt{1-x^2}\,dx \\ &= \int_0^1 \sqrt{1-x^2}\,dx. \end{aligned}$$

An antiderivative of the function $\sqrt{1-x^2}$ is obtained next by substitution

$$\begin{aligned}
\int \sqrt{1-x^2}\,dx &= \left\{ \begin{array}{c} x = \sin t \\ dx = \cos t\,dt \end{array} \right\} \\
&= \int \sqrt{1-\sin^2 t}\,\cos t\,dt \\
&= \int \cos^2 t\,dt \\
&= \int \frac{1+\cos 2t}{2}\,dt \\
&= \frac{1}{2}t + \frac{1}{4}\sin 2t + C \\
&= \frac{1}{2}t + \frac{1}{2}\sin t\cos t + C \\
&= \frac{1}{2}t + \frac{1}{2}\sin t\sqrt{1-\sin^2 t} + C \\
&= \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{(1-x^2)} + C
\end{aligned}$$

Then the average value of f on the interval $[0,1]$ is given by

$$\begin{aligned}
AV(f) &= \int_0^1 \sqrt{1-x^2}\,dx \\
&= \left(\frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{(1-x^2)}\right)\Bigg|_0^1 \\
&= \frac{\pi}{4}.
\end{aligned}$$

Example 9 *Find the average value of the function $v(t) = V_p \sin 2\pi\nu t$ on one period (cycle) and the average value of $v^2(t)$ on half a period (cycle).*

Solution The function $v(t)$ repeats itself when $v(t+T) = v(t)$,i.e.

$$2\pi\nu T = 2\pi$$

and the period is given by $T = 1/\nu$.

1. Average value of $v(t)$ on $[0,T]$

$$\begin{aligned}
AV(v(t)) &= \frac{1}{T-0}\int_0^T V_p \sin 2\pi\nu t\,dt \\
&= \frac{V_p}{T}\int_0^T \sin 2\pi\nu t\,dt \\
&= \frac{V_p}{T}\,\frac{-\cos 2\pi\nu t}{2\pi\nu}\Bigg|_0^T
\end{aligned}$$

$$
\begin{aligned}
&= \frac{V_p}{T}\frac{-\cos 2\pi\nu T - (-1)}{2\pi\nu} \\
&= \frac{V_p}{T}\frac{1-\cos 2\pi}{2\pi\nu} \\
&= 0.
\end{aligned}
$$

2. Average value of $v^2(t)$ on the interval $[0, T/2]$

$$
\begin{aligned}
AV(v^2(t)) &= \frac{1}{T/2-0}\int_0^{T/2} V_p^2 \sin^2 2\pi\nu t\, dt \\
&= \frac{2V_p^2}{T}\int_0^{T/2} \sin^2 2\pi\nu t\, dt \\
&= \frac{2V_p^2}{T}\int_0^{T/2} \frac{1-\cos 4\pi\nu t}{2}\, dt \\
&= \frac{2V_p^2}{T}\left(\frac{t}{2}+\frac{\sin 4\pi\nu t}{8\pi\nu}\right)\Bigg|_0^{T/2} \\
&= \frac{2V_p^2}{T}\left(\frac{T}{4}+\frac{\sin 2\pi\nu T}{8\pi\nu}\right) \\
&= \frac{2V_p^2}{T}\left(\frac{T}{4}\right) \\
&= \frac{V_p^2}{2}.
\end{aligned}
$$

3 Areas and Volumes

3.1 Area Between Two Curves

Let f and g be two continuous functions defined on the interval $[a, b]$ such that $f(x) \leq g(x)$. Let R the plane region enclosed by the curves $f(x)$ and $g(x)$ and the vertical lines $x = a$ and $x = b$. In order to calculate the area of the region R we start dividing the interval $[a, b]$ into n consecutive subintervals of the same length $\Delta x = (b - a)/n$ and evaluating both functions f and g at a point c_i in each subinterval. Then we draw rectangles from the curve $f(x)$ to the curve $g(x)$ with base Δx and height $g(c_i) - f(c_i)$. An approximation of the area is given by the sum

$$Area(R) \simeq \sum_{i=1}^{n} [g(c_i) - f(c_i)]\ \Delta x.$$

The limit of these sums when $n \to +\infty$ yields the exact area of the region

$$
\begin{aligned}
Area(R) &= \lim_{n\to+\infty} \sum_{i=1}^{n} [g(c_i) - f(c_i)]\ \Delta x \\
&= \int_a^b [g(x) - f(x)]\ dx.
\end{aligned}
$$

Note that the functions f and g not need be positive as long as $f(x) \leq g(x)$. If both curves intersect each other at a point c then the region should be divided into two different subregions, one where $f(x) \leq g(x)$ and another one where $g(x) \leq f(x)$. Then the area of the entire region is given by

$$Area(R) = \int_a^c [g(x) - f(x)]\ dx + \int_c^b [f(x) - g(x)]\ dx.$$

To find the area of the plane region enclosed between two curves take the following steps.

1. Find the intersection points of both curves if any.
2. Plot the curves within the limits of integration and draw representative rectangles.
3. Identify the height of the rectangles and set an integral (or integrals if the curves intersect each other within the limits of integration).
4. Evaluate the integral (or integrals)

Example 10 *Find the area of the region enclosed by the curves* $y = \sin x$ *and* $y = \sin 2x$ *between* $x = 0$ *and* $x = \pi$.

Solution

1. To find the intersection points we solve the equation $f(x) = g(x)$ or $f(x) - g(x) = 0$

$$\begin{aligned} f(x) - g(x) &= 0 \\ \sin x - \sin 2x &= 0 \\ \sin x - 2\sin x \cos x &= 0 \\ \sin x\,(1 - 2\cos x) &= 0 \end{aligned}$$

   The solutions are given by $\sin x = 0$ or $\cos x = 1/2$. Within the interval $[0, \pi]$ the solutions are $0, \pi/3$ and π.

2. Plot of the curves $f(x) = \sin x$ and $g(x) = \sin 2x$ on the interval $[0, \pi]$ and draw representative thin rectangles (the vertical lines on the plot)
3. The height of the rectangles is

$$g(x) - f(x) = \sin 2x - \sin x$$

   on the subinterval $[0, \pi/3]$ and

$$f(x) - g(x) = \sin x - \sin 2x$$

   on the subinterval $[\pi/3, \pi]$. Then the area is defined by

$$A = \int_0^{\pi/3} [g(x) - f(x)]\ dx + \int_{\pi/3}^{\pi} [f(x) - g(x)]\ dx.$$

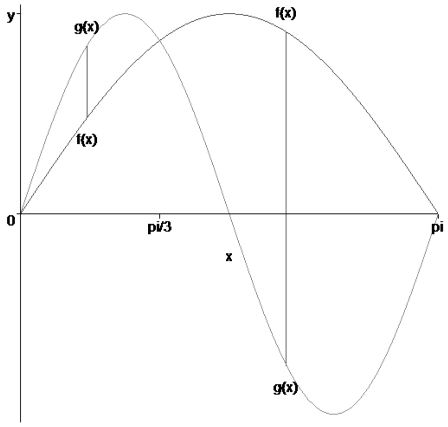


Figure 1:

4. Evaluate the integrals

$$
\begin{aligned}
A_1 &= \int_0^{\pi/3} [\sin 2x - \sin x]\ dx \\
&= \left(-\frac{1}{2}\cos 2x + \cos x\right)\Big|_0^{\pi/3} \\
&= \left[\left(-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0 + \cos 0\right)\right] \\
&= \left[\left(-\frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{2}\right) - \left(-\frac{1}{2} + 1\right)\right] \\
&= \frac{1}{4}.
\end{aligned}
$$

and

$$
\begin{aligned}
A_2 &= \int_{\pi/3}^{\pi} [\sin x - \sin 2x]\ dx \\
&= \left(-\cos x + \frac{1}{2}\cos 2x\right)\Big|_{\pi/3}^{\pi} \\
&= \left[\left(-\cos\pi + \frac{1}{2}\cos 2\pi\right) - \left(-\cos\frac{\pi}{3} + \frac{1}{2}\cos\frac{2\pi}{3}\right)\right] \\
&= \left[\left(-(-1) + \frac{1}{2}\right) - \left(-\frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)\right)\right] \\
&= \frac{9}{4}.
\end{aligned}
$$

Then the total area is

$$A = A_1 + A_2 = \frac{1}{4} + \frac{9}{4} = \frac{5}{2}.$$

3.2 Volumes of Solids of Revolution

A solid of revolution is a solid that can be generated by revolving a plane region around an axis. For example a sphere can be generated by rotating the top half of a circle around its diameter. Below we review three different methods to calculate the volume of these kinds solids.

3.2.1 Disk Method

Let's consider a solid that can be generated by revolving about the x axis the plane region enclosed by the curve $y = f(x)$, $a \leq x \leq b$, and the x axis. To calculate its volume we partition the interval the interval $[a, b]$ into n consecutive subintervals of the same length Δx and slice the solid, as we would a loaf of bread, by planes perpendicular to the x axis at the partition points. We can approximate the volume of the ith slice by the volume of the disk (circular cylinder) generated by rotating the vertical rectangle whose base is Δx and its height the value of f at a point c_i in the ith subinterval. Noting that the radius of the face of the disk is the height of the rectangle we have

$$\Delta V \simeq \pi f(c_i)^2 \, \Delta x.$$

Adding up the volumes of each disk we obtain an approximation to the total volume of the solid

$$V \simeq \sum_{i=1}^{n} \pi f(c_i)^2 \, \Delta x.$$

Taking the limit as $n \to +\infty$ we get the exact volume

$$V = \int_a^b \pi f(x)^2 \, dx.$$

Example 11 *Find the volume of a sphere of radius R.*

Solution The sphere can be generated by rotating the curve $y = \sqrt{R - x^2}$ between $x = -R$ and $x = R$. Therefore

$$\begin{aligned} V &= \int_{-R}^{R} \pi \left(\sqrt{R^2 - x^2}\right)^2 \, dx \\ &= \int_{-R}^{R} \pi \left(R^2 - x^2\right) \, dx \\ &= 2\pi \int_0^R \left(R^2 - x^2\right) \, dx \end{aligned}$$

$$
\begin{aligned}
&= 2\pi \left(R^2 x - \frac{x^3}{3} \right) \Big|_0^R \\
&= \frac{4}{3}\pi R^3.
\end{aligned}
$$

3.2.2 The Washer Method

Now we consider a solid of revolution generated by revolving about an x axis the region enclosed between an outer radius $r_e(x)$ and an inner radius $r_i(x)$, $a \leq x \leq b$.This solid has a hole in it and the cross sections perpendicular to the axis of revolution are perforated disks or washers. The volume of an approximating washer is

$$\Delta V \simeq \pi \left[r_e(c_i)^2 - r_i(c_i)^2 \right] \Delta x.$$

Adding up the volumes of each washer and taking the limit as $n \to +\infty$ we get

$$V = \int_a^b \pi \left[r_e(x)^2 - r_i(x)^2 \right] dx.$$

Example 12 *A hole of length L is drilled through the center of a sphere of radius R.Find the volume of the remaining portion of the sphere.*

Solution The outer radius is given by the circle which shapes the sphere when revolved about the x axis,

$$r_e = \sqrt{R^2 - x^2}.$$

The inner radius is the radius r of the hole. By the geometry of the sphere we have

$$\left(\frac{L}{2} \right)^2 + r^2 = R^2.$$

The the volume remaining is given by

$$
\begin{aligned}
V &= 2\int_0^{L/2} \pi \left[r_e(x)^2 - r_i(x)^2 \right] dx \\
&= 2\int_0^{L/2} \pi \left[R^2 - x^2 - r^2 \right] dx \\
&= 2\pi \int_0^{L/2} \left[\left(\frac{L}{2} \right)^2 - x^2 \right] dx \\
&= 2\pi \left[\left(\frac{L}{2} \right)^2 x - \frac{x^3}{3} \right]_0^{L/2} \\
&= \frac{\pi L^3}{6}.
\end{aligned}
$$

3.2.3 Cylindrical Shells

The above methods were based on slices perpendicular to the axis of revolution. A different approach uses slices parallel to the axis called cylindrical shells.

Let's consider a solid that can be generated by revolving about the y axis the plane region enclosed by the curve $y = f(x)$, $a \le x \le b$, and the x axis. To calculate its volume we partition the interval the interval $[a, b]$ into n consecutive subintervals of the same length Δx and slice the solid by revolving the ith rectangle about the y axis. The resulting element of volume is the solid limited by two cylinders about the y axis and its volume can be approximated by

$$\Delta V \simeq 2\pi c_i f(c_i)\,\Delta x$$

where c_i is a point in the ith subinterval. Adding up the volumes of each shell we obtain an approximation to the total volume of the solid

$$V \simeq \sum_{i=1}^{n} 2\pi c_i f(c_i)\,\Delta x.$$

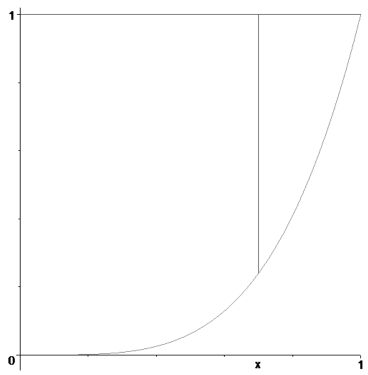
Taking the limit as $n \to +\infty$ we get the exact volume

$$V = \int_a^b 2\pi x f(x)\,dx.$$

This formula assumes $f(x) \ge 0$ and $0 \le a \le b$ but it can be applied in more general situations if we think of $f(x)$ as the height of the shells and x the distance of the shells to the axis of revolution.

Example 13 *Find the volume of the solid generated by revolving the region enclosed by the curves $y = 1$ and $y = x^4$ between $x = 0$ and $x = 1$.*

Solution In this case the height of the shells is $f(x) = 1 - x^4$ and the distance to the y axis is



Then the volume is

$$\begin{aligned} V &= \int_0^1 2\pi x \left(1 - x^4\right) \, dx \\ &= 2\pi \int_0^1 \left(x - x^5\right) \, dx \\ &= 2\pi \left(\frac{x^2}{2} - \frac{x^6}{6} \right) \Big|_0^1 \\ &= \frac{2\pi}{3}. \end{aligned}$$

4 Improper Integrals

The concept of the definite integral can be extended to include the case of integrals with infinite limits of integration and integrals of functions that become infinite at a point within the interval of integration. This is achieved by combining the definite integral defined above with the process of limit.

1. If f is continuous on $[a, +\infty)$

$$\int_a^{+\infty} f(x)\, dx = \lim_{b \to +\infty} \int_a^b f(x)\, dx.$$

2. If f is continuous on $(-\infty, b]$

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx.$$

3. If f is continuous on $(-\infty, +\infty)$

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_{-\infty}^{a} f(x)\, dx + \int_a^{+\infty} f(x)\, dx.$$

In the first two cases we say that the integrals are convergent when the limit exists and is finite, otherwise we say they are divergent. In the third case we say the integral is convergent when both integrals are convergent and divergent when one or both are divergent.

1. If f is continuous on $[a, b)$ and becomes infinite at b

$$\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx.$$

2. If f is continuous on $(a, b]$ and becomes infinite at a

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx.$$

3. If f becomes infinite at c, $a < c < b$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In the first two cases we say that the integrals are convergent when the limit exists and is finite, otherwise we say they are divergent. In the third case we say the integral is convergent when both integrals are convergent and divergent when one or both are divergent.

Example 14 *Evaluate the following integral*

$$\int_0^{+\infty} e^{-x}\,dx.$$

Solution Integral with infinite limit of integration

$$\begin{aligned}\int_0^{+\infty} e^{-x}\,dx &= \lim_{b\to+\infty}\int_0^b e^{-x}\,dx \\ &= \lim_{b\to+\infty}\left(-e^{-x}\right)\Big|_0^b \\ &= \lim_{b\to+\infty}\left(1-e^{-b}\right) \\ &= 1.\end{aligned}$$

The integral is convergent and its value is 1.

Example 15 *Analyze the convergence of the integral*

$$\int_0^{+\infty} e^{-x^2}\,dx.$$

Solution Integral with infinite limit of integration. First we decompose the integral into two integrals

$$\int_0^{+\infty} e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^{+\infty} e^{-x^2}\,dx$$

The integrand function satisfies

$$\begin{aligned}0 &\le e^{-x^2} \le 1 \text{ if } 0 \le x \le 1 \\ 0 &\le e^{-x^2} \le e^{-x} \text{ if } 1 \le x \le 1\end{aligned}$$

Therefore

$$\begin{aligned}0 &\le \int_0^{+\infty} e^{-x^2}\,dx \le \int_0^1 e^{-x^2}\,dx + \int_1^{+\infty} e^{-x^2}\,dx \\ &\le \int_0^1 1\,dx + \int_1^{+\infty} e^{-x}\,dx \\ &= 1 + e^{-1}.\end{aligned}$$

We conclude that the integral is convergent and its value is between 0 and $1 + e^{-1}$.

Example 16 *Evaluate the integral*

$$\int_0^1 \frac{dx}{x}.$$

Solution Improper integral with function becoming infinite at the left end of the interval of integration.

$$\begin{aligned} \int_0^1 \frac{dx}{x} &= \lim_{c \to 0^+} \int_c^1 \frac{dx}{x} \\ &= \lim_{c \to 0^+} (\ln x)\Big|_c^1 \\ &= \lim_{c \to 0^+} (-\ln c) \\ &= +\infty. \end{aligned}$$

The integral is divergent.