Applications of Derivatives

1 Introduction

This section is a review of derivatives with an emphasis on applications. We will start reviewing the concept of derivative of a function $y = f(x)$ defined on an interval (a, b) at a point x_0 in that interval.

Let $\Delta x = h$ be an increment of the independent variable x, i.e. the change in the value of x from x_0 to $x_1 = x_0 + h$, and let Δy be the corresponding change in the value of the dependent variable y,i.e.

$$\Delta y = f(x_0 + h) - f(x_0).$$

Then the derivative of f at x_0 is defined by

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \end{aligned}$$

if the limit exists.

2 Tangent and Normal Lines

The *tangent line* to the graph of the function f at a point $P(x_0, f(x_0))$ is the line through the point P having slope $m = f'(x_0)$, i.e. the line of equation

$$y - f(x_0) = f'(x_0)(x - x_0).$$

The *normal line* to the graph of the function f at a point $P(x_0, f(x_0))$ is the line through the point P having slope $m = -1/f'(x_0)$, i.e. the line of equation

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0).$$

Example 1 *A train moves down a rail track at night. The track has the shape of the curve $y = 2/x$, $x > 0$. At what point does the headlight beam hit the x-axis when the train is at the point* $(2, 1)$.

Solution Derivative of the function $y = 2/x$

$$y' = -\frac{2}{x^2}.$$

Slope of the tangent line at the point $(2, 1)$

$$m = y'(2) = -\frac{2}{2^2} = -\frac{1}{2}.$$

Equation of the tangent line at that point

$$y - 1 = -\frac{1}{2}(x - 2)$$

or

$$y = 2 - \frac{x}{2}.$$

This line crosses the x-axis at

$$0 = 2 - \frac{x}{2}$$

i.e.

$$x = 4.$$

3 Differentials and Linear Approximation

The tangent line approximates the graph of function near the point of tangency. We can observe this by plotting the graph of the function together with the tangent line at a point and zooming in.

The function

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

whose graph is the tangent line to the graph of the function f at the point $P(x_0, f(x_0))$ is called the linear approximation of f at x_0 and $f(x) \simeq L(x)$ near x_0.

Changes in the value of a function can be approximated by the changes in the linear approximation. This is done via differentials.

We define the differential of the independent variable x by

$$dx = \Delta x = h$$

and the differential of the dependent variable y by

$$dy = f'(x_0)\, dx.$$

This differential can be used to approximate the change in $y = f(x)$

$$\begin{aligned} \Delta y &= f(x_0 + h) - f(x_0) \\ &\simeq L(x_0 + h) - f(x_0) \\ &= f'(x_0)\, h \\ &= dy. \end{aligned}$$

In other words, the absolute change in the value of the function f corresponding to a change Δx, from x_0 to $x_1 = x_0 + h$, can be estimated by dy, i.e.

$$\Delta f = \Delta y \simeq dy = f'(x_0)\, dx.$$

The relative change in the value of the function $y = f(x)$ can be estimated by

$$\frac{\Delta f}{f} \simeq \frac{dy}{y} = \frac{f'(x_0)\, dx}{f(x_0)}$$

and the percentage change by

$$100\frac{\Delta f}{f} \simeq 100\frac{dy}{y} = 100\frac{f'(x_0)\, dx}{f(x_0)}.$$

Example 2 *Estimate the maximum variation that can be tolerated on the diagonal of a 16:9 TV screen if the area of the screen is to be within* 0.1% *of its intended size. Estimate the effect that an error in the measurement of the diagonal of* $\pm 0.05cm$ *has on the area of a* $125cm$ *diagonal screen size.*

Solution

1. Let x be the width of the screen, y the height and D the length of the diagonal. The area of the screen is given by

   $$A = xy.$$

   Since

   $$\frac{x}{y} = \frac{16}{9}$$

   we have

   $$A = xy = \frac{9}{16}x^2.$$

   Finally using the relationship

   $$\begin{aligned} D^2 &= x^2 + y^2 \\ &= \left(1 + \frac{9^2}{16^2}\right) x^2 \end{aligned}$$

   we obtain

   $$A = kD^2 \qquad \text{with} \qquad k = \frac{16 \times 9}{16^2 + 9^2} = 0.4273.$$

   Since

   $$\frac{\Delta A}{A} \simeq \frac{dA}{A} = \frac{2kDdD}{kD^2} = 2\frac{dD}{D}$$

   the deviation in the diagonal should be

   $$2\frac{dD}{D} \leq 0.001$$

i.e.

$$\frac{dD}{D} \leq 0.0005 = 0.05\%.$$

2. Area of a $125cm$ diagonal screen

$$A = kD^2 = (0.4273)(125^2) = 6676.6cm^2.$$

Estimate of the error in the area due to an error of $\pm 0.05cm$ in the diagonal

$$\begin{aligned} \Delta A &\simeq dA \\ &= 2kDdD \\ &= \pm 2(0.4273)(125)(0.05) \\ &= \pm 5.34cm^2. \end{aligned}$$

4 Optimization Problems

This is one of the most important applications of derivatives. In an optimization problem you are asked to maximize/minimize a certain variable that depends on the value of another variable. To set up and solve these problems follow the following steps:

1. Try to understand the problem and draw a diagram representing the situation.
2. Identify the given and the unknown magnitudes.
3. List all the relations between the unknown variables.
4. Select the variable to maximize/minimize and using the equations listed before write a function that relates that variable to only one of the other variables. This may require some work to eliminate other extra variables.
5. Identify a domain for the independent variable. At this point your problem should read like this

$$\text{Maximize/minimize } y = f(x), \qquad x \in D.$$

6. Solving this max/min problem is straightforward if you can plot a graph of the function f. If not,
7. Find the critical points of f. These are the solutions to the equation

$$f'(x) = 0$$

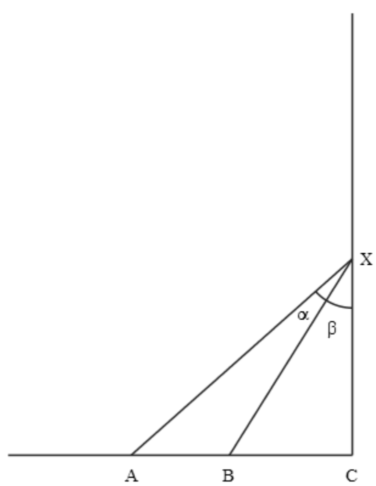
and those points where f does not have a derivative.

8. Create a list with the values of f at the critical points and the values f at the endpoints of the interval D. Select the maximum and the minimum value from this list. If the function is not defined at an endpoint a instead of the value $f(a)$ calculate the $\lim_{x\to a} f(x)$.

9. Do not forget to check the consistency of your results with the actual nature of your problem.

Example 3 *During a soccer match a winger is running with the ball close to the sideline. At what distance x from the corner flag should the player kick the ball to see the goal with the widest angle possible? Assume that the field is w meters wide and the goal p meters wide.*

Solution



First we need to express the angle α to be maximized as a function of the distance x. From the above figure we have

$$\tan(\alpha+\beta) = \frac{AC}{XC} = \frac{w+p}{2x}$$

and

$$\tan\beta = \frac{BC}{XC} = \frac{w-p}{2x}.$$

Therefore

$$\begin{aligned} \alpha &= (\alpha+\beta)-\beta \\ &= \arctan\frac{w+p}{2x} - \arctan\frac{w-p}{2x}. \end{aligned}$$

The critical points of this function are the solutions of the equation $\alpha' = 0$, i.e.

$$\alpha' = \frac{-\dfrac{w+p}{2x^2}}{1+\left(\dfrac{w+p}{2x}\right)^2} - \frac{-\dfrac{w-p}{2x^2}}{1+\left(\dfrac{w-p}{2x}\right)^2} = 0.$$

Simplifying, first we obtain

$$\frac{(w+p)}{4x^2+(w+p)^2} - \frac{(w-p)}{4x^2+(w-p)^2} = 0.$$

then

$$(w+p)\left[4x^2+(w-p)^2\right] - (w+p)\left[4x^2+(w-p)^2\right] = 0$$

and finally

$$4x^2 = w^2 - p^2.$$

Therefore the only critical point in the region $x > 0$ is

$$x_0 = \frac{1}{2}\sqrt{w^2-p^2}.$$

Since $\alpha(x) > 0$ and both

$$\lim_{x\to+\infty} \alpha(x) = 0 \text{ and } \lim_{x\to 0^+} \alpha(x) = \frac{\pi}{2} \text{-} \frac{\pi}{2} = 0$$

α is maximum at x_0

$$\alpha_{\max} = \alpha(x_0) = \arctan\sqrt{\frac{w+p}{w-p}} - \arctan\sqrt{\frac{w-p}{w+p}}.$$