Curves

1 Parametrized Curves

When a particle moves in a plane its position at an instant t can be determined by two coordinates $x(t)$ and $y(t)$ that depend on t. With these two functions we can describe the path traced by the particle as t changes.

Given two continuous functions $x(t)$ and $y(t)$ we say that the set of points $P = (x, y)$ defined by

$$x = x(t), \qquad y = y(t), \qquad a \leq t \leq b.$$

is a **curve** in the plane. The functions $x(t)$ and $y(t)$ are called the **parametric equations** of the curve and the variable t is called a **parameter** for the curve.

The parameter t does not always denote time. It can denote an angle, the distance travelled by the particle along its path or something else.

If $\mathbf{r}$ is the **position vector** of the particle, i.e. the vector from the origin O to the point $P = (x(t), y(t))$, we can write

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, \qquad a \leq t \leq b,$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit coordinate vectors. This equation defines a function that assigns to each t on the interval $[a, b]$ a vector of two components and is called a **vector function of a real variable**.

In the three dimensional space, position is determined by three coordinates x, y and z. So a curve in space is defined by a set of three parametric equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad a \leq t \leq b,$$

or in vector notation by a three component vector function

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \qquad a \leq t \leq b.$$

Example 1 *The parametric equations of a circle of radius R centered at the origin are*

$$x = R\cos t, \qquad y = R\sin t, \qquad 0 \leq t \leq 2\pi.$$

Solution The parameter represents the angle that the radius OP makes with the positive x-axis. Squaring both equations and adding them up we obtain

$$\begin{aligned} x^2 + y^2 &= R^2 \cos^2 t + R^2 \sin^2 t \\ &= R^2 \left(\cos^2 t + \sin^2 t\right) \\ &= R^2 \end{aligned}$$

which is the equation of a circle of radius R centered at the origin.

If we observe the position as t increases

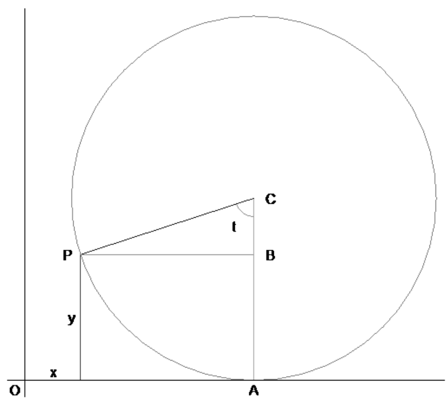
t	0	$\pi/2$	π	$3\pi/2$	2π
(x, y)	$(1, 0)$	$(0, 1)$	$(-1, 0)$	$(0, -1)$	$(1, 0)$

we conclude that the circle is traced counterclockwise exactly once.

Example 2 *A cycloid is the path traced by a point on the circumference of a wheel rolling along a straight line. If R is the radius of the wheel the parametric equations for the cycloid are*

$$x = R(t - \sin t), \qquad y = R(1 - \cos t), \qquad t \geq 0.$$

Solution Set the straight line to be the x-axis and start with the point P at the origin. Assume that the wheel is rolling right. After the wheel has turned an angle t we have, see figure,



$$x = OA - PB, \qquad y = AC - BC.$$

Since

$$\begin{aligned} OA &= arc(PA) = Rt \\ PB &= R \sin t \\ BC &= R \cos t \\ AC &= R \end{aligned}$$

we obtain the parametric equations of the cycloid

$$x = Rt - R\sin t, \qquad y = R - R\cos t, \qquad t \geq 0.$$

Example 3 *A helix in the* 3*D-space*

$$x = \cos t, \qquad y = \sin t, \qquad z = t, \qquad -\infty < t < +\infty.$$

Solution Since

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

the projection of the points onto the plane $z = 0$ lie on the unit circle. As t increases z also increases so the resulting curve has the shape of a vine winding around a circular pillar (cylinder). This curve is called a helix and every time t increases by 2π the helix completes one turn around the cylinder.

Tangent Vector The **derivative** of a vector function

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

is defined by

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

where the limit is given by

$$\mathbf{r}'(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}.$$

Since the vector $\mathbf{r}(t)$ represents the position of a particle in motion the vector $\mathbf{v} = \mathbf{r}'(t)$ represents the **velocity** of the particle. When different from $\mathbf{0}$, the vector $\mathbf{r}'(t)$ defines a **tangent vector** to the curve indicating the direction of motion. The parametric equations of the **tangent line** to the curve $\mathbf{r}(t)$ at $t = t_0$ are given by

$$x = x(t_0) + x'(t_0)\lambda, \qquad y = y(t_0) + y'(t_0)\lambda,$$

where λ is the parameter of the line. By eliminating the parameter λ, the tangent line can be written in normal form by

$$\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)}.$$

Example 4 *Find the tangent line to the ellipse*

$$\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j}, \qquad 0 \leq t \leq 2\pi,$$

at $t = \pi/6$.

Solution Derivative of the vector function $\mathbf{r}(t)$

$$\mathbf{r}'(t) = (-3\sin t)\,\mathbf{i} + (2\cos t)\,\mathbf{j}.$$

Tangent vector at $t = \pi/6$

$$\begin{aligned}
\mathbf{r}'\left(\frac{\pi}{6}\right) &= \left(-3\sin\frac{\pi}{6}\right)\mathbf{i} + \left(2\cos\frac{\pi}{6}\right)\mathbf{j} \\
&= \left(-3\frac{1}{2}\right)\mathbf{i} + \left(2\frac{\sqrt{3}}{2}\right)\mathbf{j} \\
&= -\frac{3}{2}\,\mathbf{i} + \sqrt{3}\,\mathbf{j}.
\end{aligned}$$

Position at $t = \pi/6$

$$\begin{aligned}
\mathbf{r}\left(\frac{\pi}{6}\right) &= \left(3\cos\frac{\pi}{6}\right)\mathbf{i} + \left(2\sin\frac{\pi}{6}\right)\mathbf{j} \\
&= \left(3\frac{\sqrt{3}}{2}\right)\mathbf{i} + \left(2\frac{1}{2}\right)\mathbf{j} \\
&= \frac{3\sqrt{3}}{2}\,\mathbf{i} + \mathbf{j}.
\end{aligned}$$

Parametric equations of the tangent line

$$x = \frac{3\sqrt{3}}{2} - \frac{3}{2}\lambda, \qquad y = 1 + \sqrt{3}\lambda$$

Implicit equation of the tangent line

$$\frac{x - \frac{3\sqrt{3}}{2}}{-\frac{3}{2}} = \frac{y-1}{\sqrt{3}}$$

or

$$2\sqrt{3}x + 3y = 12.$$

Length of a Curve Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, a \leq t \leq b$, be a plane curve that is traced exactly once as t increases from a to b. To find the length of this curve we partition the interval $[a, b]$ into n small subintervals of length Δt. This partition of the parameter interval breaks the curve down into small arcs. If Δs represents the length of one these arcs we have

$$\|\Delta s\| \simeq \|\Delta \mathbf{r}(t)\| \simeq \|\mathbf{r}'(t)\|\,\Delta t.$$

This suggest to define the length of the curve from a to b by the integral

$$L = \int_a^b \|\mathbf{r}'(t)\|\;dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\;dt.$$

The norm of the velocity vector $\mathbf{v} = \mathbf{r}'(t)$ is called **speed**

$$Speed = \frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

where s is the arc length from the initial point $P(a) = (x(a), y(a))$ up to a generic point $P(t) = (x(t), y(t))$.

In the case of a space curve we add a third coordinate to the above formulas

$$L = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt.$$

Example 5 *Calculate the length of the cycloid*

$$x = R(t - \sin t), \qquad y = R(1 - \cos t), \qquad 0 \le t \le 2\pi.$$

Solution Position vector

$$\mathbf{r}(t) = R(t - \sin t)\,\mathbf{i} + R(1 - \cos t)\,\mathbf{j}.$$

Derivative of the position vector or velocity or

$$\mathbf{r}'(t) = R(1 - \cos t)\,\mathbf{i} + R \sin t\,\mathbf{j}.$$

Speed

$$\begin{aligned}
\frac{ds}{dt} &= \|\mathbf{r}'(t)\| \\
&= \sqrt{R^2(1 - \cos t)^2 + R^2 \sin^2 t} \\
&= R\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\
&= R\sqrt{2(1 - \cos t)} \\
&= R\sqrt{4 \sin^2 \frac{t}{2}} \\
&= 2R \sin \frac{t}{2}.
\end{aligned}$$

Length

$$\begin{aligned}
L &= \int_0^{2\pi} \|\mathbf{r}'(t)\| \, dt \\
&= \int_0^{2\pi} 2R \sin \frac{t}{2} \, dt \\
&= 2R \left. \frac{-\cos \frac{t}{2}}{\frac{1}{2}} \right|_0^{2\pi} \\
&= 8R.
\end{aligned}$$

The Area of a Surface of Revolution When we revolve a curve around axis we get a surface of revolution. Let $\mathbf{r}(t) = x(t)\,\mathbf{i}+y(t)\,\mathbf{j}$, $a \leq t \leq b$, be a plane curve that is traced exactly once as t increases from a to b. As before we partition the interval $[a, b]$ into n small subintervals of length Δt. When the curve is revolved around the x-axis each resulting arc of length Δs generates a band of the surface whose area is

$$\Delta S \simeq 2\pi y \Delta s \simeq 2\pi y \left\|\mathbf{r}'(t)\right\| \Delta t.$$

This suggest to define the **area of the surface of revolution around the x-axis** from a to b by the integral

$$S = \int_a^b 2\pi y(t) \left\|\mathbf{r}'(t)\right\|\, dt = \int_a^b 2\pi y(t) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

In the same way we define the **area of the surface of revolution around the y-axis** from a to b by the integral

$$S = \int_a^b 2\pi x(t) \left\|\mathbf{r}'(t)\right\|\, dt = \int_a^b 2\pi x(t) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

Example 6 *Area of the surface of revolution generated by revolving the circle*

$$x = R + r\cos t, \qquad y = r\sin t, \qquad 0 \leq t \leq 2\pi,$$

around the y-axis.

Solution

$$\begin{aligned} S &= \int_a^b 2\pi x(t) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \\ &= \int_0^{2\pi} 2\pi(R + r\cos t) \sqrt{[-r\sin t]^2 + [r\cos t]^2}\, dt \\ &= 2\pi r \int_0^{2\pi} (R + r\cos t) \sqrt{\sin^2 t + \cos^2 t}\, dt \\ &= 2\pi r \int_0^{2\pi} (R + r\cos t)\, dt \\ &= 2\pi r\, (Rt + r\sin t)\big|_0^{2\pi} \\ &= 4\pi^2 Rr. \end{aligned}$$

2 Polar Coordinates

A point P in the plane can be located using a pair of **polar coordinates** (r, θ) called **radius** and **polar angle** respectively. To define this pair of coordinates we use a reference system consisting of a point O, called **pole**, and a ray from O called **polar axis**. Then,

- r is the signed distance from the point P to the pole, and
- θ is the directed angle from the polar axis to the ray OP.

With this definition a point can be represented with infinitely many pairs of polar coordinates. The pairs (r, θ) and $(-r, \theta + \pi)$ represent the same point. If (r, θ) is a pair of polar coordinates of a point P the pairs $(r, \theta \pm 2k\pi)$, k integer, also represent the same point P.The pole itself can be represented by any pair of the form $(0, \theta)$ for any angle θ.

If a unique representation is needed r is limited to positive values, $r \geq 0$, and θ to the interval $[0, 2\pi)$ or alternatively $(-\pi, \pi]$.

Relationship Between Cartesian and Polar Coordinates When polar and Cartesian coordinates are used at the same time we take the origin of both systems together and we make the polar axis coincide with the positive x-axis. Then the following equations hold:

- To convert from polar to Cartesian coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta$$

- To convert from Cartesian to polar coordinates

$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x.$$

Curves in Polar Coordinates The representation of a function $r = r(\theta)$, $\alpha \leq \theta \leq \beta$, in the polar plane is a curve in polar coordinates. For instance an equation of the form $r = k$ represents a circle of radius k centered at the origin. An equation of the form $\theta = k$ represents a line through the origin with slope $\tan k$.

To graph a function $r = r(\theta)$ in polar coordinates we start calculating a sufficient number of pairs (r, θ) and then plot the curve by connecting all the corresponding points in the polar plane.

Parametric equations of a polar curve $r = r(\theta)$, $\alpha \leq \theta \leq \beta$, are given by

$$x = r(\theta)\cos\theta, \qquad y = r(\theta)\sin\theta, \qquad \alpha \leq \theta \leq \beta.$$

Slope of the Tangent Line A tangent vector at the point $\theta = \theta_0$ is

$$x' = r'(\theta_0)\cos\theta_0 - r(\theta_0)\sin\theta_0, \qquad y' = r'(\theta_0)\sin\theta_0 + r(\theta_0)\cos\theta_0,$$

and the slope of the tangent line if $x'(\theta_0) \neq 0$ is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\theta_0} = \frac{y'(\theta_0)}{x'(\theta_0)} = \frac{r'(\theta_0)\sin\theta_0 + r(\theta_0)\cos\theta_0}{r'(\theta_0)\cos\theta_0 - r(\theta_0)\sin\theta_0}.$$

If the curve goes through the origin at $\theta = \theta_0$ then the slope of the tangent line is

$$m = \tan\theta_0.$$

Area in Polar Coordinates Consider a function $r = r(\theta)$, $\alpha \leq \theta \leq \beta$ and let P be the point of polar coordinates $(r(\theta), \theta)$. As θ varies between α and β the point P traces the curve $r = r(\theta)$ and the ray OP sweeps the fan-shaped region enclosed by the curve and the rays $\theta = \alpha$ and $\theta = \beta$. In order to find the area of this region we divide the interval $\alpha \leq \theta \leq \beta$ into n subintervals and we choose an angle θ_i in each subinterval. Then the area swept by the ray OP when θ varies in each subinterval can be approximated by the area of the circular sector of radius $r(\theta_i)$ and central angle $\Delta\theta_i$ and the total area by

$$A \simeq \sum_{i=1}^{n} \frac{1}{2} r^2(\theta_i)\, \Delta\theta_i.$$

In the limit we can define

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2(\theta)\, d\theta_i.$$