CALCULUS 2F3M-I

7/4/2012

Duration: 80 minutes

Exercise 1 *Determine whether the series is convergent or divergent. If the series is convergent find its sum.*

1. (a) $\sum_{n=0}^{\infty} \frac{n!}{10^n}$

   (b) $\sum_{n=1}^{\infty} \frac{3}{2} \frac{4^n}{5^{n-1}}$

Solution:

1. (a) Applying the ratio test

$$\begin{aligned} L &= \lim_{n\to+\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to+\infty} \frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}} \\ &= \lim_{n\to+\infty} \frac{(n+1)!}{n!} \frac{10^n}{10^{n+1}} \\ &= \lim_{n\to+\infty} \frac{(n+1)}{10} \\ &= +\infty \end{aligned}$$

   Since $L > 1$ the series is divergent.

   (b) The series can be written in the form

$$\sum_{n=1}^{\infty} \frac{3}{2} \frac{4^n}{5^{n-1}} = \sum_{n=1}^{\infty} \frac{3 \times 4}{2} \frac{4^{n-1}}{5^{n-1}} = \sum_{n=1}^{\infty} 6 \left(\frac{4}{5}\right)^{n-1} = \sum_{n=0}^{\infty} 6 \left(\frac{4}{5}\right)^{n}$$

   Therefore the series is a geometric series of ratio $r = 4/5$. Since $|r| < 1$ the series is convergent and its sum is

$$S = 6 \frac{1}{1-r} = 6 \frac{1}{1-\frac{4}{5}} = 30.$$

Exercise 2 *Find the length of arc of the curve defined by the parametric equations*

$$x = 2t^2, \ y = 2t^3$$

from $t = 1$ *to* $t = 2$.

Solution:
The length of arc is given by

$$\begin{aligned}
L &= \int_1^2 \sqrt{x'(t)^2 + y'(t)^2}\, dt \\
&= \int_1^2 \sqrt{(4t)^2 + (6t^2)^2}\, dt \\
&= \int_1^2 \sqrt{16t^2 + 36t^4}\, dt \\
&= \int_1^2 2t\sqrt{4 + 9t^2}\, dt \\
&= \left. \frac{(4+9t^2)^{3/2}}{\frac{3}{2}9} \right|_1^2 \\
&= \frac{2}{27}\left(40^{3/2} - 13^{3/2}\right) \\
&= \frac{160\sqrt{10}}{27} - \frac{26\sqrt{13}}{27}.
\end{aligned}$$

Exercise 3 *Find the extrema of the function*

$$f(x, y) = x^2 y$$

subject to the constraint

$$x^2 + 8y^2 = 24.$$

Solution:
Using the Lagrange method we define

$$L(x, y) = x^2 y + \lambda(x^2 + 8y^2 - 24).$$

The extrema of f subject to the above condition are critical points of L. These critical points are the solutions of the system

$$\begin{aligned}
L_x &= 2xy + 2\lambda x = 0 \\
L_y &= x^2 + 16\lambda y = 0 \\
L_\lambda &= x^2 + 8y^2 - 24 = 0.
\end{aligned}$$

From the first equation we obtain

$$x = 0 \text{ or } \lambda = -y.$$

For $x = 0$ the second equation yields

$$y = 0 \text{ or } \lambda = 0.$$

The point $x = 0, y = 0$ does not meet the third equation and for $\lambda = 0$ the third condition equation produces the solutions

$$x = 0, \;\; y = \pm\sqrt{3}.$$

For $\lambda = -y$ the second equation yields

$$y = \pm\frac{1}{4}x.$$

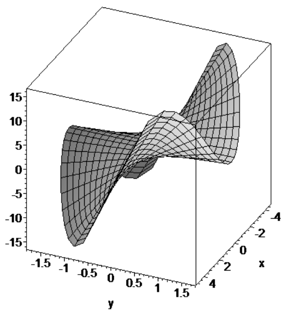
This together with the third equation produces the points

$$x = \pm 4, \;\; y = \pm 1.$$

Now we calculate the values of f at each point and we have

$$f(0, \pm\sqrt{3}) = 0; \;\; f(\pm 4, 1) = 16; \;\; f(\pm 4, -1) = -16.$$

Therefore an absolute maximum is reached at the points $(4, 1)$ and $(-4, 1)$ and a minimum at the points $(4, -1)$ and $(-4, -1)$.



Exercise 4 *Evaluate*

$$\iint_D xy\, dxdy$$

if D is the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Solution:

$$\begin{aligned} \iint_D xy\, dxdy &= \int_0^1 \left(\int_{x^2}^{\sqrt{x}} xy\, dy \right) dx \\ &= \int_0^1 \left(x \frac{y^2}{2} \Big|_{x^2}^{\sqrt{x}} \right) dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{1}{2}\left(x^2 - x^5\right)\, dx \\
&= \left.\frac{x^3}{6} - \frac{x^6}{12}\right|_0^1 \\
&= \frac{1}{12}.
\end{aligned}
$$