CALCULUS 2F2M-I

7/12/2013

Duration: 80 minutes

Exercise 1 *A square has sides of length 2 units. Starting with this square we construct a new square by drawing line segments through the midpoints of the sides of the first square. Then a third square is constructed by drawing line segments through the midpoints of the sides of the second square. If this procedure is applied to each resulting square for an unlimited number of times, what is the total perimeter of all the squares formed?*

Solution If l_n is the side of the n-th square the side of the $(n+1)$-th square is

$$l_{n+1} = \sqrt{\left(\frac{l_n}{2}\right)^2 + \left(\frac{l_n}{2}\right)^2} = \frac{l_n}{\sqrt{2}} = l_n\frac{\sqrt{2}}{2}.$$

Therefore, the sum of the perimeters of all squares is the infinite series

$$4l_0 + 4l_0\frac{\sqrt{2}}{2} + 4l_0\left(\frac{\sqrt{2}}{2}\right)^2 + \cdots = 4l_0\sum_{n=0}^{\infty}\left(\frac{\sqrt{2}}{2}\right)^n$$

where $l_0 = 2$ is the side of the initial square.

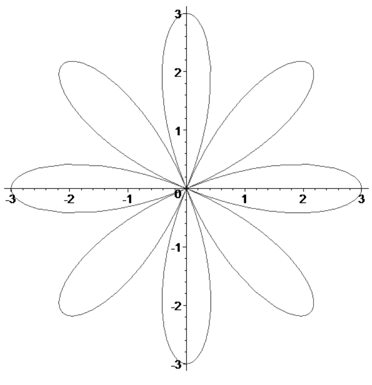
The series is a geometric series of ratio $r = \sqrt{2}/2$. Since $|r| < 1$ the series is convergent and its sum is

$$P = \frac{4l_0}{1 - \frac{\sqrt{2}}{2}} = \frac{16}{2 - \sqrt{2}} = 8\left(2 + \sqrt{2}\right).$$

Exercise 2 *Find the area of one leaf of the rose*

$$r = 3\cos 4\theta.$$

Solution Sketch of the rose between $\theta = 0$ and $\theta = 2\pi$



Each leaf is sketched between two consecutive zeros of the function $r = 3\cos 4\theta$. In the interval $[0, 2\pi]$ the solutions to the equation $r = 0$ are

$$4\theta = \frac{\pi}{2}(2k+1), \qquad k = 0, 1, \cdots, 7$$

or

$$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \cdots \frac{15\pi}{8}.$$

The area enclosed by the rose is

$$A_{rose} = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta$$

and the area of one leaf is

$$A_{leaf} = \frac{1}{2}\int_{\pi/8}^{3\pi/8} r^2 \, d\theta = \frac{1}{8}A_{rose}.$$

Therefore

$$\begin{aligned} A_{rose} &= \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 9\cos^2 4\theta \, d\theta \\ &= \frac{9}{2}\int_0^{2\pi} \frac{1+\cos 8\theta}{2} \, d\theta \\ &= \frac{9}{4}\int_0^{2\pi} d\theta + \frac{9}{4}\int_0^{2\pi} \cos 8\theta \, d\theta \\ &= \frac{9}{4}\left(\theta\right)\Big|_0^{2\pi} + \frac{9}{4}\left.\frac{\sin 8\theta}{8}\right|_0^{2\pi} \\ &= \frac{9\pi}{2} \end{aligned}$$

and

$$A_{leaf} = \frac{9\pi}{16}.$$

Exercise 3 *Show that the surfaces*

$$36x^2 + 4y^2 + 9z^2 = 108$$

and

$$xyz = 6$$

are tangent at the point $(1, 3, 2)$.

Solution We have to show that both surfaces have the same tangent plane at the given point or to show that their normal vectors are proportional.

Let

$$F(x, y, z) = 36x^2 + 4y^2 + 9z^2 - 108$$

and

$$G(x, y, z) = xyz - 6.$$

A normal vector to the surface $F(x, y, z) = 0$ is given by

$$\nabla F = 72x\,\mathbf{i} + 8y\,\mathbf{j} + 18z\,\mathbf{k}$$

and a normal vector to the surface $G(x, y, z) = 0$ is given by

$$\nabla G = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}.$$

At the point $(1, 3, 2)$

$$\nabla F(1, 3, 2) = 72\,\mathbf{i} + 24\,\mathbf{j} + 36\,\mathbf{k}$$

and

$$\nabla G(1, 3, 2) = 6\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}.$$

Since

$$\nabla F(1, 3, 2) = 12\nabla G(1, 3, 2)$$

both surfaces are tangent at the point $(1, 3, 2)$ and share the same tangent plane

$$6(x - 1) + 2(y - 3) + 3(z - 2) = 0$$

or

$$6x + 2y + 3z = 18.$$

Exercise 4 *Find the volume under the plane*

$$z = 4x + y$$

and above the region D enclosed by the parabolas $y = x^2$ and $y = 4x - x^2$.

Solution The volume is given by the integral

$$V = \iint_D (4x + y)\,dxdy.$$

The intersection points of the parabolas are the solutions of the equation

$$x^2 = 4x - x^2 \quad \text{or} \quad 2x(x - 2) = 0$$

i.e. $x = 0$ and $x = 2$.Then

$$
\begin{aligned}
V &= \int_0^2 \left(\int_{x^2}^{4x-x^2} (4x + y)\, dy \right) dx \\
&= \int_0^2 \left((4xy + \frac{1}{2}y^2) \Big|_{x^2}^{4x-x^2} \right) dx \\
&= \int_0^2 \left[4x \left(4x - x^2\right) + \frac{1}{2} \left(4x - x^2\right)^2 - 4x^3 - \frac{1}{2}x^4 \right] dx \\
&= \int_0^2 \left[-12x^3 + 24x^2 \right] dx \\
&= \left(-3x^4 + 8x^3 \right) \Big|_0^2 \\
&= 16.
\end{aligned}
$$