CALCULUS

End-term Exam
May 29, 2018
Duration: 45m

Exercise 1 *An astronaut is travelling along the path*

$$\mathbf{r}(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 3t^3\,\mathbf{k}.$$

At $t = 1$ he/she shuts off the engine and coasts along the tangent line.

1. *What distance has the astronaut travelled between $t = 0$ and $t = 1$?*
2. *Will the astronaut pass through the point $(4, 9, 12)$ after shutting off the engine?*

Solution

1. The derivative (velocity) is given by

$$\mathbf{r}'(t) = 2\,\mathbf{i} + 6t\,\mathbf{j} + 9t^2\,\mathbf{k}$$

and the speed is

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{2^2 + (6t)^2 + (9t^2)^2} \\ &= \sqrt{4 + 36t^2 + 81t^4} \\ &= \sqrt{\left(2 + 9t^2\right)^2} \\ &= 2 + 9t^2. \end{aligned}$$

The distance travelled is the integral of the speed

$$\begin{aligned} L &= \int_0^1 \|\mathbf{r}'(t)\|\; dt \\ &= \int_0^1 \left(2 + 9t^2\right)\; dt \\ &= \left(2t + 3t^3\right)\Big|_0^1 \\ &= 5. \end{aligned}$$

2. Position of the astronaut at $t = 1$

$$\mathbf{r}(1) = 2\,\mathbf{i} + 3\,\mathbf{j} + 3\,\mathbf{k}.$$

Tangent vector at $t = 1$

$$\mathbf{r}'(1) = 2\,\mathbf{i}{+}6\,\mathbf{j}{+}9\,\mathbf{k}.$$

Therefore the equation of the tangent line is given by

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}(1) + \mathbf{r}'(1)t \\ &= (2+2t)\,\mathbf{i} + (\mathbf{3}{+}6t)\,\mathbf{j} + (\mathbf{3}{+}9t)\,\mathbf{k}\end{aligned}$$

or in continuous form

$$\frac{x-2}{2} = \frac{y-3}{6} = \frac{z-3}{9}.$$

This line goes trough the point $(4, 9, 12)$ because

$$\frac{4-2}{2} = \frac{9-3}{6} = \frac{12-3}{9}.$$

Exercise 2 *Let f be the function*

$$f(x, y) = e^{2x+3y}.$$

1. *Find the tangent plane to the surface $z = f(x, y)$ at the point $(0, 0)$.*
2. *Calculate the double integral*

$$\iint_D f(x, y)\, dA.$$

where D is the region enclosed by the lines $y = -2x/3, y = x$ and $x = 1$.

Solution

1. The tangent plane at $(0, 0)$ is given by

$$z = f(0, 0) + f_x(0, 0)(x-0) + f_y(0, 0)(y-0)$$

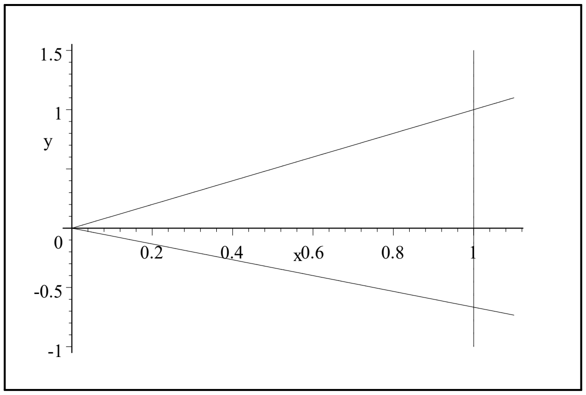
Since the partial derivatives are

$$\begin{aligned}f_x &= 2e^{2x+y} \\ f_y &= 3e^{2x+y}\end{aligned}$$

at $(0, 0)$ we have

$$z{=}1 + 2x + 3y.$$

2. Region D



The double integral is equal to the following iterated integral

$$
\begin{aligned}
\iint_D e^{2x+3y}\,dA &= \int_0^1 \left(\int_{-2x/3}^{x} e^{2x+3y}\,dy \right) dx \\
&= \int_0^1 e^{2x} \left(\int_{-2x/3}^{x} e^{3y}\,dy \right) dx - \\
&= \int_0^1 e^{2x} \left(\frac{e^{3y}}{3} \right) \Big|_{-2x/3}^{x} dx - \\
&= \int_0^1 e^{2x} \left(\frac{e^{3x} - e^{-2x}}{3} \right) dx \\
&= \frac{1}{3} \int_0^1 \left(e^{5x} - 1 \right) dx \\
&= \frac{1}{3} \left(\frac{e^{5x}}{5} - x \right) \Big|_0^1 \\
&= \frac{e^5 - 6}{15}.
\end{aligned}
$$