# CALCULUS

End-term Exam
May 30, 2017
Duration: 45m

**Exercise 1** *Given the curve of parametric equations*

$$x = 3t^2; \quad y = t^3 - 3t; \qquad -2 \leq t \leq 2$$

*find*

1. *The length of the loop that starts and ends at the point* $(9, 0)$.
2. *The point(s) where the curve has a horizontal tangent.*

**Solution:**

1. First we need to find the values of $t$ such that $x = 9$ and $y = 0$.These are the common solutions of the equations $3t^2 = 9$ and $t^3 - 3t = 0$, i.e. $t = \pm\sqrt{3}$. The length of the loop is given by the integral of the speed

$$L = \int_{-\sqrt{3}}^{+\sqrt{3}} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

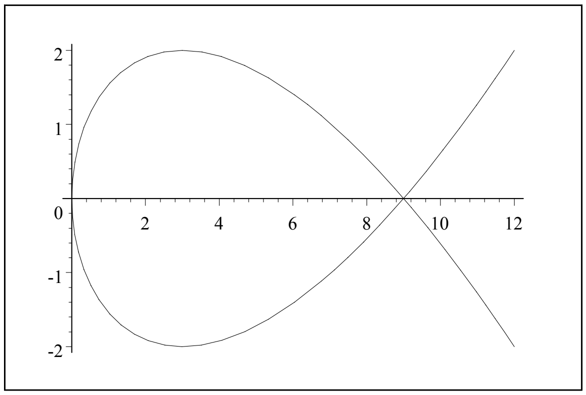
The derivatives are $x'(t) = 6t$ and $y'(t) = 3t^2 - 3$. Therefore,

$$\begin{aligned} speed &= \sqrt{[x'(t)]^2 + [y'(t)]}^2 \\ &= \sqrt{36t^2 + 9(t^2 - 1)^2} \\ &= \sqrt{9\left(t^2 + 1\right)^2} \\ &= 3(t^2 + 1) \end{aligned}$$

and

$$\begin{aligned} L &= 3\int_{-\sqrt{3}}^{+\sqrt{3}} (t^2 + 1)\, dt \\ &= 6\int_{0}^{+\sqrt{3}} (t^2 + 1)\, dt \\ &= 6\left(\frac{t^3}{3} + t\right)\Bigg|_0^{\sqrt{3}} \\ &= 12\sqrt{3}. \end{aligned}$$

2. The points where the tangent is horizontal are determined by the condition $y'(t) = 3t^2 - 3 = 0$. This implies $t = \pm 1$ so $x = 3$ and $y = \pm 2$. Plot of the curve




**Exercise 2** *Consider the function*

$$f(x,y) = 2x^2 + 3y^2.$$

1. *Find the directional derivative of $f$ at the point $(1/2, -1/2)$ in the direction of the vector $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$.*
2. *Evaluate the double integral of $f$ on the region $D$ enclosed by the lines $x = -1$, $y = -1$ and $x + y = 1$.*

**Solution:**

1. Partial derivatives

$$\begin{aligned} f_x &= 4x \\ f_y &= 6y \end{aligned}$$

Gradient vector at the point $(1/2, -1/2)$

$$\boldsymbol{\nabla} f(1/2, -1/2) = 2\mathbf{i} - 3\mathbf{j}.$$

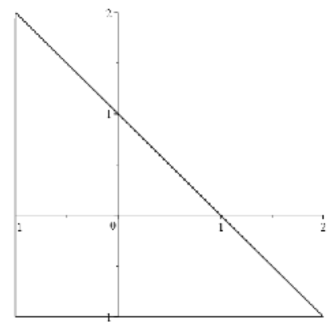
Unit vector $\mathbf{d}$ in the direction of $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$

$$\mathbf{d} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{4\mathbf{i} - 3\mathbf{j}}{\sqrt{4^2 + 3^2}} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}.$$

Directional derivative of $f$ at $(1/2, -1/2)$ in the direction of vector $\mathbf{u}$

$$\begin{aligned} D_{\mathbf{u}}(f)(1/2, -1/2) &= \boldsymbol{\nabla} f(1/2, -1/2) \cdot \mathbf{d} \\ &= (2\mathbf{i} - 3\mathbf{j}) \cdot \left(\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}\right) \\ &= 2\frac{4}{5} + (-3)\left(-\frac{3}{5}\right) \\ &= \frac{17}{5}. \end{aligned}$$

2. Region $D$



Double integral

$$
\begin{aligned}
\iint_D \left(2x^2+3y^2\right)\, dA &= \int_{-1}^{2}\left(\int_{-1}^{1-x}\left(2x^2+3y^2\right)\, dy\right) dx \\
&= \int_{-1}^{2}\left[\left(2x^2y+y^3\right)\Big|_{-1}^{1-x}\right] dx \\
&= -\int_{-1}^{2}\left(2-3x+7x^2-3x^3\right) dx \\
&= \left(2x-3\frac{x^2}{2}+7\frac{x^3}{3}-3\frac{x^4}{4}\right)\Bigg|_{-1}^{2} \\
&= \frac{45}{4}.
\end{aligned}
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