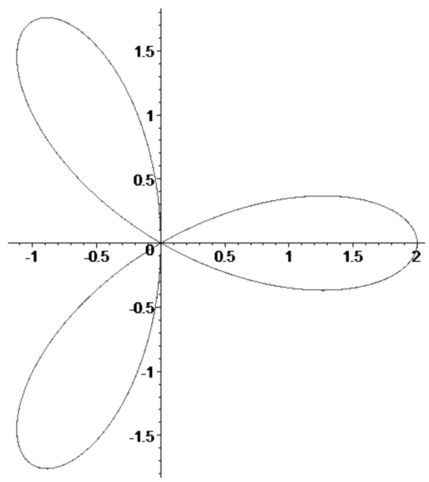
CALCULUS

End-term Exam
May 31, 2016
Duration: 45m

Exercise 1 *Find the area of a petal of the rose given by the polar equation* $r = 2\cos(3\theta)$.

Solution: Plot of the rose



At $\theta = 0$, the radius is $r = 2$. Then the radius becomes zero at $\theta = \pi/6$. Therefore the area of one petal is given by

$$\begin{aligned} A &= 2\int_0^{\pi/6} \frac{1}{2} r^2 \, d\theta \\ &= 4\int_0^{\pi/6} \cos^2(3\theta)\, d\theta \\ &= 4\int_0^{\pi/6} \frac{1+\cos(6\theta)}{2}\, d\theta \\ &= 2\int_0^{\pi/6} d\theta + 2\int_0^{\pi/6} \cos(6\theta)\, d\theta \end{aligned}$$

If $0 \le \theta \le \pi/6$ then $0 \le 6\theta \le \pi$ and the second integral becomes zero. Therefore

$$A = \frac{\pi}{3}.$$

Exercise 2 *Find the rate of change of the function* $f(x,y,z) = 3x^2 + 6y^2 - 4z^2 + 2xy$ *at the point* $P = (1,2,-1)$ *in the direction* $\mathbf{d} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution: The rate of change is given by the directional derivative of f at P in the direction of the unit vector $\mathbf{u} = \mathbf{d}/\left\|\mathbf{d}\right\|$, i.e.

$$D_{\mathbf{u}}(f)(P) = \nabla f(P) \cdot \mathbf{u}.$$

The gradient of f is

$$\nabla f = (6x + 2y)\mathbf{i} + (12y + 2x)\mathbf{j} - 8z\mathbf{k}.$$

At $P = (1, 2, -1)$

$$\nabla f(P) = 10\mathbf{i} + 26\mathbf{j}{+}8\mathbf{k}.$$

Since the norm of $\mathbf{d}$ is

$$\|\mathbf{d}\| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$$

we have

$$\mathbf{u} = \frac{\mathbf{d}}{\|\mathbf{d}\|} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

and

$$\begin{aligned} D_{\mathbf{u}}(f)(P) &= (10\mathbf{i} + 26\mathbf{j}{+}8\mathbf{k})\cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) \\ &= 10\frac{2}{3} + 26\frac{1}{3} + 8\left(-\frac{2}{3}\right) \\ &= 10. \end{aligned}$$

Exercise 3 *Evaluate the double integral*

$$\iint\limits_R 3xy\, dxdy$$

where R is the triangle formed by the y-axis and the lines $y = 2$ and $y = x$.

Solution: The lines $y = 2$ and $y = x$ cross each other at the point $(2, 2)$. By iteration

$$\begin{aligned} \iint_R 3xy\, dxdy &= \int_0^2 \left(\int_x^2 3xy\, dy\right) dx \\ &= \int_0^2 3x \left(\frac{y^2}{2}\right)\Big|_x^2 dx \\ &= \int_0^2 3x\left(2 - \frac{x^2}{2}\right) dx \\ &= \int_0^2 \left(6x - \frac{3x^3}{2}\right) dx \\ &= \left(3x^2 - \frac{3x^4}{8}\right)\Big|_0^2 \\ &= 6. \end{aligned}$$