CALCULUS

End-term Exam
June 4, 2014
Duration: 45m

Exercise 1 *Given the curve of parametric equations*

$$x = e^{3t/4} - t^2; \quad y = t + e^{-3t/4}; \qquad -2 \le t \le 4$$

find

1. *The tangent line at* $t_0 = 1$.
2. *The point(s) where the curve has a horizontal tangent.*

Solution

1. Derivatives of the parametric equations

$$\begin{aligned} x'(t) &= \frac{3}{4}e^{3t/4} - 2t \\ y'(t) &= 1 - \frac{3}{4}e^{-3t/4} \end{aligned}$$

Direction of the tangent line at $t_0 = 1$

$$\begin{aligned} x'(1) &= \frac{3}{4}e^{3/4} - 2 \simeq -0.412 \\ y'(1) &= 1 - \frac{3}{4}e^{-3/4} \simeq 0.645 \end{aligned}$$

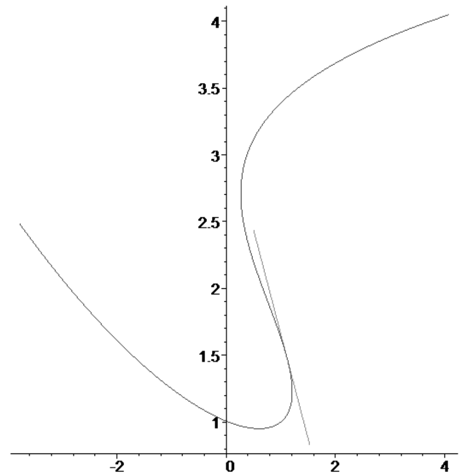
Position at $t_0 = 1$

$$\begin{aligned} x(1) &= e^{3/4} - 1 \simeq 1.118 \\ y(1) &= 1 + e^{-3/4} \simeq 1.472 \end{aligned}$$

Parametric equations of the tangent line at $t_0 = 1$

$$\begin{aligned} x(t) &= x(1) + x'(1)t = 1.118 - 0.412t \\ y(t) &= y(1) + y'(1)t = 1.472 + 0.645t \end{aligned}$$

Plot of the curve and the tangent line



2. To find the point with horizontal tangent we need to solve the equation

$$y'(t) = 1 - \frac{3}{4}e^{-3t/4} = 0.$$

The solution is

$$t_h = -\frac{4}{3}\ln\frac{4}{3} \simeq -0.384.$$

The coordinates of the point with horizontal tangent are

$$\begin{aligned} x(t_h) &\simeq 0.603 \\ y(t_h) &\simeq 0.950. \end{aligned}$$

Exercise 2 *Consider the function*

$$f(x,y) = e^{2x-y}$$

on the region D enclosed by the parabola $y = x^2 - 1$ and the line $y = x + 1$. Calculate:

1. *The directional derivative of f at the point $(0,0)$ in the direction of the vector $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$.*

2. *The double integral*

$$\iint_D f(x,y)\,dA.$$

Solution

1. Partial derivatives

$$\begin{aligned} f_x &= 2e^{2x-y} \\ f_y &= -e^{2x-y} \end{aligned}$$

Gradient vector at $(0,0)$

$$\nabla f(0,0) = 2\mathbf{i} - \mathbf{j}.$$

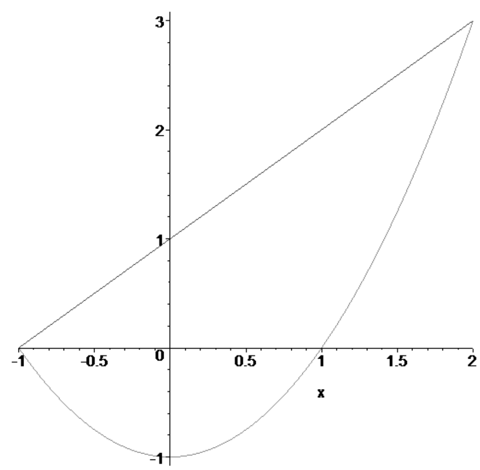
Unit vector in the direction of $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

Directional derivative of f at $(0,0)$ in the direction of vector $\mathbf{u}$

$$\begin{aligned} D_{\mathbf{u}}(f)(0,0) &= \nabla f(0,0) \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|} \\ &= (2\mathbf{i} - \mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) \\ &= 2\frac{3}{5} + (-1)\left(-\frac{4}{5}\right) \\ &= 2. \end{aligned}$$

2. Region D



The double integral is equal to the following iterated integral which is evaluated with Maple

$$\begin{aligned} \iint_D e^{2x-y}\, dA &= \int_{-1}^{2} \int_{x^2-1}^{x+1} e^{2x-y}\, dy dx \\ &= -e + \frac{1}{2}\sqrt{\pi}e^2 \operatorname{erf}(1) + e^{-2} + \frac{1}{2}\sqrt{\pi}e^2 \operatorname{erf}(2). \end{aligned}$$

The integral is given in terms of the error function. Approximate value

$$\iint_D e^{2x-y}\,dA \simeq 9.453.$$