CALCULUS

Final Exam
June 6, 2013
Duration: 60m

Exercise 1 *Consider the curve given by the parametric equations*

$$x = e^{-t}\cos t, \quad y = e^{-t}\sin t, \quad 0 \le t \le 2\pi.$$

Find:

1. *The points at which the curve has horizontal and vertical lines.*
2. *Parametric equations of the tangent line to the curve at $t = \pi$.*
3. *The length of arc between $t = 0$ and $t = \pi$.*

Solution

1. The components of a tangent vector to the curve at instant t are

$$x' = -e^{-t}\cos t - e^{-t}\sin t, \quad y' = -e^{-t}\sin t + e^{-t}\cos t.$$

This vector will be horizontal when $y' = 0$, i.e. when

$$-e^{-t}\sin t + e^{-t}\cos t = e^{-t}(\cos t - \sin t) = 0 \quad \text{or} \quad \cos t = \sin t.$$

The solutions to this equation on the interval $0 \le t \le 2$ are $t = \pi/4$ and $t = 5\pi/4$ which correspond to the points

$$\left(\frac{e^{-\pi/4}\sqrt{2}}{2}, \frac{e^{-\pi/4}\sqrt{2}}{2}\right) \quad \text{and} \quad \left(-\frac{e^{-5\pi/4}\sqrt{2}}{2}, -\frac{e^{-5\pi/4}\sqrt{2}}{2}\right).$$

The tangent vector will be vertical when $x' = 0$, i.e. when

$$-e^{-t}\cos t - e^{-t}\sin t = -e^{-t}(\cos t + \sin t) = 0 \quad \text{or} \quad \cos t = -\sin t.$$

The solutions to this equation on the interval $0 \le t \le 2$ are $t = 3\pi/4$ and $t = 7\pi/4$ which correspond to the points

$$\left(-\frac{e^{-3\pi/4}\sqrt{2}}{2}, \frac{e^{-3\pi/4}\sqrt{2}}{2}\right) \quad \text{and} \quad \left(\frac{e^{-7\pi/4}\sqrt{2}}{2}, -\frac{e^{-7\pi/4}\sqrt{2}}{2}\right).$$

2. The components of the tangent vector at $t = \pi$ are

$$x'(\pi) = e^{-\pi}, \quad y'(\pi) = -e^{-\pi}.$$

The position is the point

$$x(\pi) = -e^{-\pi}, \quad y(\pi) = 0.$$

The parametric equations of the tangent line at $t = \pi$ are

$$\begin{aligned} x(t) &= x(\pi) + x'(\pi)t = (-1+t)e^{-\pi} \\ y(t) &= y(\pi) + y'(\pi)t = te^{-\pi}. \end{aligned}$$

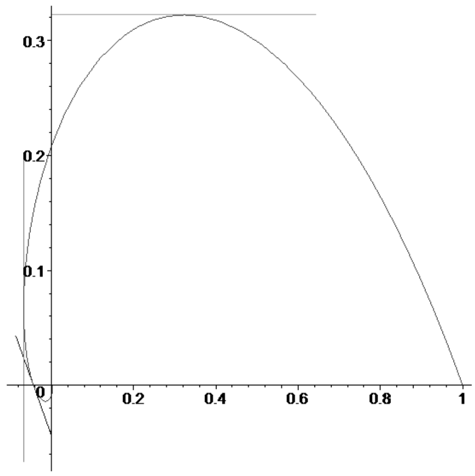
3. The speed is the norm of the tangent vector

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{[x'(t)]^2 + [y'(t)]^2} \\ &= \sqrt{[-e^{-t}\cos t - e^{-t}\sin t]^2 + [-e^{-t}\sin t + e^{-t}\cos t]^2} \\ &= e^{-t}\sqrt{\left[\cos^2 t + 2\cos t \sin t + \sin^2 t\right] + \left[\sin^2 t - 2\sin t\cos t + \cos^2 t\right]} \\ &= \sqrt{2}e^{-t}. \end{aligned}$$

The arc length is given by the integral of the speed

$$\begin{aligned} L &= \int_0^{\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \\ &= \int_0^{\pi} \sqrt{2}e^{-t}\, dt \\ &= \sqrt{2}\left(-e^{-t}\right)\Big|_0^{\pi} \\ &= \sqrt{2}\left(1 - e^{-\pi}\right). \end{aligned}$$

4. Plot of the curve, horizontal tangent at $t = \pi/4$, vertical tangent at $t = 3\pi/4$ and the tangent line at $t = \pi$



Exercise 2 *Given the function*

$$f(x, y, z) = 3x^2 + y^2 + 4z^2$$

and the point $P_0 = (1, 1, 1)$, find:

1. *The directions in which the function increases and decreases most rapidly at the point P_0 and the rates of change in these directions.*
2. *The rate of change in the direction $\mathbf{d} = 2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$.*
3. *The tangent plane to the level surface*

$$3x^2 + y^2 + 4z^2 = 8$$

at the point P_0.

Solution

1. Gradient of the function

$$\boldsymbol{\nabla} f = 6x\,\mathbf{i} + 2y\,\mathbf{j} + 8z\,\mathbf{k}$$

Gradient at the point $P_0 = (1, 1, 1)$

$$\boldsymbol{\nabla} f(1, 1, 1) = 6\,\mathbf{i} + 2\,\mathbf{j} + 8\,\mathbf{k}.$$

Norm of the gradient

$$\|\boldsymbol{\nabla} f(1, 1, 1)\| = \sqrt{6^2 + 2^2 + 8^2} = 2\sqrt{26}.$$

Direction in which the function increases most rapidly at P_0

$$\mathbf{u} = \frac{\boldsymbol{\nabla} f(1, 1, 1)}{\|\boldsymbol{\nabla} f(1, 1, 1)\|} = \frac{3}{\sqrt{26}}\,\mathbf{i} + \frac{1}{\sqrt{26}}\,\mathbf{j} + \frac{4}{\sqrt{26}}\,\mathbf{k}.$$

Greatest rate of increase

$$\|\boldsymbol{\nabla} f(1, 1, 1)\| = 2\sqrt{26}.$$

Direction in which the function decreases most rapidly at P_0

$$\mathbf{u} = -\frac{\boldsymbol{\nabla} f(1, 1, 1)}{\|\boldsymbol{\nabla} f(1, 1, 1)\|} = -\frac{3}{\sqrt{26}}\,\mathbf{i} - \frac{1}{\sqrt{26}}\,\mathbf{j}\tilde{n}\frac{4}{\sqrt{26}}\,\mathbf{k}.$$

Greatest rate of decrease

$$\|\boldsymbol{\nabla} f(1, 1, 1)\| = -2\sqrt{26}.$$

2. Unit vector in the direction of $\mathbf{d} = 2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$

$$\mathbf{u} = \frac{\mathbf{d}}{\|\mathbf{d}\|} = \frac{2}{3}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} - \frac{2}{3}\,\mathbf{k}.$$

Rate of change of f at P_0 in this direction

$$\begin{aligned} D_{\mathbf{u}}f(1,1,1) &= \boldsymbol{\nabla} f(1,1,1)\cdot\mathbf{u} \\ &= [6\,\mathbf{i} + 2\,\mathbf{j} + 8\,\mathbf{k}]\cdot\left[\frac{2}{3}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} - \frac{2}{3}\,\mathbf{k}\right] \\ &= 4 + \frac{2}{3} - \frac{16}{3} \\ &= -\frac{2}{3}. \end{aligned}$$

3. The surface

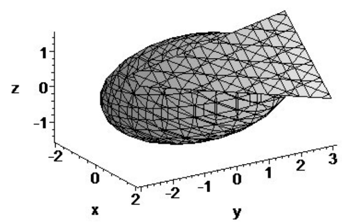
$$3x^2 + y^2 + 4z^2 = 8$$

is the level surface of the function f that goes trough the point P_0. The gradient vector $\boldsymbol{\nabla} f(1,1,1)$ is normal to the surface at P_0 therefore the equation of the tangent plane at P_0 is given by

$$6(x-1) + 2(y-1) + 8(z-1) = 0$$

or

$$3x + y + 4z = 8.$$



Example 3 *You want to build a rectangular crate with a fixed volume $V = 9$ cubic meters using three different material for each opposite sides. If the cost of each material is $a = 1/2$, $c = 1/3$ and $c = 1/4$ euro per square meter, find the dimensions of the most economical crate and its total cost.*

Solution The cost of a rectangular crate of dimensions x, y, z is given by

$$C = 2ayz + 2bxz + 2cxy$$

and the volume by

$$V = xyz.$$

Therefore we state the following Max-min problem: Find the extrema of the function

$$f(x, y, z) = \frac{C}{2} = ayz + bxz + cxy$$

subject to the condition

$$xyz - V = 0$$

and $x > 0$, $y > 0$, $z > 0$.

The Lagrange function is

$$L(x, y, z, \lambda) = ayz + bxz + cxy - \lambda\,(xyz - V)\,.$$

To find the critical points of L we calculate its partial derivatives, set them to zero and solve the resulting system

$$\begin{aligned} L_x &= bz + cy - \lambda yz = 0 \\ L_y &= az + cx - \lambda xz = 0 \\ L_z &= ay + bx - \lambda xy = 0. \\ L_\lambda &= xyz - V = 0. \end{aligned}$$

Multiplying the first equation by x, the second by y and the third one by z we obtain

$$\begin{aligned} bxz + cxy - \lambda xyz &= 0 \\ ayz + cxy - \lambda xyz &= 0 \\ ayz + bxz - \lambda xyz &= 0. \\ xyz - V &= 0. \end{aligned}$$

Subtracting the third equation from the first one and then from the second one we get

$$cx = az \quad \text{and} \quad cy = bz$$

or

$$x = \frac{a}{c} z \quad \text{and} \quad y = \frac{b}{c} z$$

Substituting these equations in the volume contraint yields

$$V = \frac{ab}{c^2} z^3 = \frac{abc}{c^3} z^3.$$

Solving for z

$$z = c \sqrt[3]{\frac{V}{abc}}$$

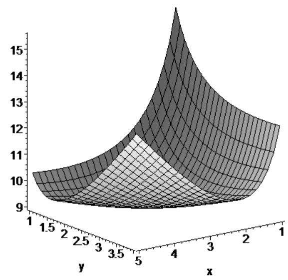


Figure 1:

and finally

$$x = a\sqrt[3]{\frac{V}{abc}} \quad \text{and} \quad y = b\sqrt[3]{\frac{V}{abc}}.$$

For these dimensions the cost is given by

$$C = 6abc\left(\frac{V}{abc}\right)^{2/3} = 6(abc)^{1/3}V^{2/3}.$$

Using the values $a = 1/2$, $b = 1/3$, $c = 1/4$, $V = 9$

$$x = 3,\ y = 2,\ z = 3/2 \quad \text{and} \quad C = 9.$$

We can plot the cost function solving the constraint for z and substituting in C to obtain

$$C = \frac{2aV}{x} + \frac{2bV}{y} + 2cxy.$$

For the particular values of the constants

$$C = \frac{9}{x} + \frac{6}{y} + \frac{1}{2}xy$$

whose plot isWe can also show the contour levels and gradient field to show that the critical point produces a minimum.