CALCULUS 2F3M-I

6/8/2012

Final Exam

Exercise 1 *Find the tangent lines to the curve*

$$x = t^3 - 3t^2 + 2t, \ \ y = t^2 - 2t + 1, \ \ -\infty < t < \infty$$

at the point $(0, 1)$. *Find the points with a vertical tangent. Plot the curve between* $t = -2$ *and* $t = 4$.

Solution:

The values of t such that $x = 0$ and $y = 1$ are the common solutions to the equations

$$\begin{aligned} t^3 - 3t^2 + 2t &= 0 \\ t^2 - 2t + 1 &= 1 \end{aligned}$$

These values are $t = 0$ and $t = 2$. The first equation has another root, $t = 1$, but this value correspond to the point $x = 0, y = 0$.

A tangent vector to the curve is given by

$$\begin{aligned} x'(t) &= 3t^2 - 6t + 2 \\ y'(t) &= 2t - 2 \end{aligned}$$

Therefore a direction of the tangent the line when $t = 0$ is $[2, -2]$ or $[1, -1]$and when $t = 2$ is $[2, 2]$ or $[1, 1]$. Then the parametric equations of the tangent lines at the point $(0, 1)$ are

$$\begin{aligned} x &= t \\ y &= 1 - t \end{aligned}$$

and

$$\begin{aligned} x &= t \\ y &= 1 + t \end{aligned}$$

Eliminating the parameter t we obtain

$$y = 1 - x$$

and

$$y = 1 + x.$$

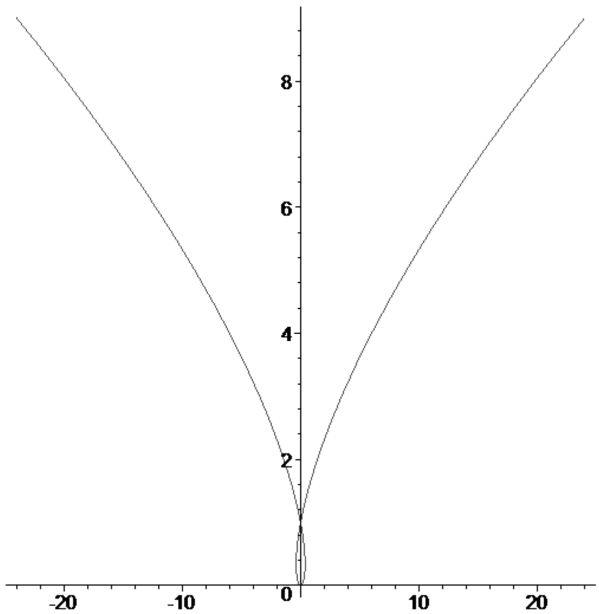
The points at which the curve has vertical tangents are the ones corresponding to the solutions of the equation

$$x'(t) = 3t^2 - 6t + 2 = 0$$

The roots are: $t = 1 + \frac{1}{3}\sqrt{3}$ and $t = 1 - \frac{1}{3}\sqrt{3}$, which correspond to the points

$$\left(\pm\frac{2\sqrt{3}}{9}, \frac{1}{3}\right).$$

Plot of the curve on the range $t = -2$, $t = 4$



Exercise 2 *The shape of a mountain can be modeled by the equation*

$$z = 1200 - 3x^2 - 2y^2$$

where distance is measured in meters, the x-axis point to the east and the y-axis to the north. A mountain climber is at the point corresponding to $x_0 = -10$, $y_0 = 5$.

1. *What is the direction of steepest ascent? and what is the rate of change in that direction?*
2. *If the climber moves in the southwest direction is he or she ascending or descending? and what is the rate?*
3. *In which direction is the climber travelling a level path? what is the equation of this path?*

Solution:

1. The direction of steepest ascent is given by the gradient vector at the point.

$$\begin{aligned}\nabla f(-10,5) &= f_x(-10,5)\mathbf{i} + f_y(-10,5)\mathbf{j} \\ &= -6x|_{(-10,5)}\,\mathbf{i} + -4y|_{(-10,5)}\,\mathbf{j} \\ &= 60\mathbf{i} - 20\mathbf{j}\end{aligned}$$

   and the rate of change the length of this vector, i.e.

$$\|\nabla f(-10,5)\| = \sqrt{60^2 + (-20)^2} = 20\sqrt{10}.$$

2. A unit vector in the southwest direction is $\mathbf{u} = (-\mathbf{i} - \mathbf{j})/\sqrt{2}$. The directional derivative is

$$\begin{aligned}D_{\mathbf{u}} f(-10,5) &= \nabla f(-10,5) \cdot \mathbf{u} \\ &= (60\mathbf{i} - 20\mathbf{j}) \cdot \left(-\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right) \\ &= -\frac{40}{\sqrt{2}}\end{aligned}$$

   Therefore the climber is descending and the rate is $\frac{40}{\sqrt{2}}$.

3. The direction of the level path is perpendicular to the gradient, i.e. $\mathbf{v}$=20$\mathbf{i}$+ 60$\mathbf{j}$. Since

$$z(-10,5) = 850$$

   the equation of the level path is

$$850 = 1200 - 3x^2 - 2y^2$$

   or

$$3x^2 + 2y^2 = 350.$$

Exercise 3 *Find an equation for the tangent plane and scalar parametric equations for the normal line to the surface*

$$z = x^2 + xy + y^2 - 4x + y - 2$$

at the point $(2, 1, -2)$ *on the surface.*

Solution: First write the equation of the surface as an implicit equation $F(x, y, z) = 0$

$$x^2 + xy + y^2 - 4x + y - z - 2 = 0.$$

Gradient of F at a general point (x, y, z)

$$\nabla F(x, y, z) = (2x + y - 4)\,\mathbf{i} + (x + 2y + 1)\,\mathbf{j} - \mathbf{k}.$$

Gradient of F at the point $(2, 1, -2)$

$$\boldsymbol{\nabla} F(2,1,-2) = \mathbf{i} + 5\,\mathbf{j} - \mathbf{k}.$$

Tangent plane

$$(x-2) + 5(y-1) - (z+2) = 0$$

or

$$x + 5y - z = 9.$$

Normal line

$$\frac{x-2}{1} = \frac{y-1}{5} = \frac{z+2}{-1}$$

or

$$\begin{aligned} x &= 2 + t \\ y &= 1 + 5t \\ z &= -2 - t. \end{aligned}$$

Exercise 4 *In order to be accepted for mailing postal regulations require that the length plus the girth (perimeter of a cross section) of a package has to be less than* 1 *meter. What are the dimensions of the rectangular box of maximum volume that can be mailed?*

Solution: Let x and y be the dimensions of the cross section and z the third dimension. Then we have to maximize the function

$$V = xyz$$

subject to the condition

$$2x + 2y + z = 1.$$

Lagrange function

$$L = xyz - \lambda(2x + 2y + z - 1)$$

The critical points of L are the solutions of the system

$$\begin{aligned} L_x &= yz - 2\lambda = 0 \\ Ly &= xz - 2\lambda = 0 \\ L_z &= xy - \lambda = 0 \\ L_\lambda &= 2x + 2y + z - 1 = 0. \end{aligned}$$

Multiplying the first equation by x, the second by y and the third by z we get

$$\begin{aligned} xyz - 2\lambda x &= 0 \\ xyz - 2\lambda y &= 0 \\ xyz - \lambda z &= 0 \\ 2x + 2y + z - 1 &= 0. \end{aligned}$$

From the first three equations we get

$$2x = 2y = z.$$

Substitution into the last equation yields

$$3z - 1 = 0$$

Therefore,

$$x = \frac{1}{6}, y = \frac{1}{6}, z = \frac{1}{3}.$$

Exercise 5 *Evaluate*

$$\iint_D x^2 y \, dx dy$$

with D the triangle whose vertices are the points $(0,0)$, $(1,2)$ and $(0,2)$.

Solution:

$$\begin{aligned}
\iint_D x^2 y \, dx dy &= \int_0^1 \left(\int_{2x}^2 x^2 y \, dy \right) dx \\
&= \int_0^1 \left. \left(x^2 \frac{y^2}{2} \right) \right|_{2x}^{2} dx \\
&= \int_0^1 x^2 (2 - 2x^2) \, dx \\
&= 2 \int_0^1 \left(x^2 - x^4 \right) dx \\
&= 2 \left. \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \right|_0^1 \\
&= \frac{4}{5}.
\end{aligned}$$